Math 308G/H - Winter 2018
Midterm 1
2018-01-27

KEY

Question	Points	Score
1	12	
2	12	
3	12	
4	12	
5	12	
Total:	60	

- There are 5 problems on this exam. Be sure you have all 5 problems on your exam.
- The final answer must be left in exact form. Box your final answer.
- You are allowed the TI-30XIIS calculator. It is possible to complete the exam without a calculator.
- You are allowed a single sheet of 2-sided handwritten self-written notes.
- You must show your work to receive full credit. A correct answer with no supporting work will receive a zero.
- Use the backsides if you need extra space. Make a note of this if you do.
- Do not cheat. This exam should represent your own work. If you are caught cheating, I will report you to the Community Standards and Student Conduct office.

Conventions:

- I will often denote the zero vector by 0.
- When I define a variable, it is defined for that whole question. The A defined in Question 2 is the same for each part.
- I often use x to denote the vector $(x_1, x_2, \ldots, x_n)$. It should be clear from context.
- Sometimes I write vectors as a row and sometimes as a column. The following are the same to me.

$$(1,2,3) \quad \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}.$$

- I write the evaluation of linear transforms in a few ways. The following are the same to me.

$$T(1,2,3) \quad T((1,2,3)) \quad T\left(\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}\right)$$

1. Give an example of each of the following. If it is not possible, write "NOT POSSIBLE".

(a) (3 points) Give an example of a linear system with more equations than variables and infinitely many solutions.

Solution:

$$\begin{aligned} x + y &= 1 \\ 2x + 2y &= 2 \end{aligned}$$

(b) (3 points) Give an example of an invertible matrix A such that A^2 is the zero matrix.

Solution: "NOT POSSIBLE". Since A is invertible so is $A^2 = A \cdot A$ so A^2 cannot be the zero matrix.

(c) (3 points) Give an example of a linear transformation $T : \mathbb{R}^3 \to \mathbb{R}^3$ that is one-to-one but $(1, 0, 1)$ is not in range(T).

Solution: "NOT POSSIBLE". Any one-to-one linear transformation from $\mathbb{R}^3$ to $\mathbb{R}^3$ must be onto.

(d) (3 points) Give an example of a linear system whose solution space is the intersection of the $w + x + y + z = 2$ and the $x + y = 1$ plane in $\mathbb{R}^4$.

Solution:

$$\begin{aligned} w + x + y + z &= 2 \\ x + y &= 1 \end{aligned}$$

2. Let

$$A = \begin{bmatrix} 1 & 0 & -1 \\ 2 & 1 & -1 \\ 0 & 1 & 2 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 2 \\ 0 & -1 \\ 0 & 0 \end{bmatrix}, \quad D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}.$$

Do the following parts:

(a) (3 points) Compute A^{-1}.

Solution:

$$\begin{bmatrix} 3 & -1 & 1 \\ -4 & 2 & -1 \\ 2 & -1 & 1 \end{bmatrix}$$

(b) (3 points) Compute AB.

Solution:

$$\begin{bmatrix} 1 & 2 \\ 2 & 3 \\ 0 & -1 \end{bmatrix}$$

(c) (3 points) Compute AD^2.

Solution:

$$\begin{bmatrix} 1 & 0 & -9 \\ 2 & 4 & -9 \\ 0 & 4 & 18 \end{bmatrix}$$

(d) (3 points) Give the general solution to $Ax = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$. (Hint: Use the first part and the fact $Ax = b \implies x = A^{-1}b$.)

Solution: Since A is invertible, there is only one solution and that solution is given by

$$A^{-1} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ -3 \\ 2 \end{bmatrix}.$$

3. Let

$$A = \begin{bmatrix} 1 & -1 & 1 & 1 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 1 & 2 \end{bmatrix}.$$

Let a_1, a_2, a_3, a_4 denote the columns of A. Let $S = \{a_1, a_2, a_3, a_4\}$. Do the following parts:

(a) (3 points) Write null(A) as a span of some vectors.

Solution: The reduced echelon form is

$$\begin{bmatrix} 1 & -1 & 0 & -1 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

The nullspace is the span of $\{(1,1,0,0), (1,0,-2,1)\}$.

(b) (3 points) Give the general solution to $Ax = a_1 + a_3$.

Solution: A particular solution is $(1,0,1,0)$. The homogeneous solution is $s_1(1,1,0,0) + s_2(1,0,-2,1)$. The general solution is then

$$(1,0,1,0) + s_1(1,1,0,0) + s_2(1,0,-2,1).$$

(c) (3 points) Is S a spanning set? If not, how many additional vectors must be added to S to make it spanning?

Solution: No. The reduced echelon form has a row of zeros. The span is at least 2 dimensional since there are 2 nonparallel columns of A. This means only 1 additional vector is required.

(d) (3 points) Let $T : \mathbb{R}^4 \to \mathbb{R}^3$ have the property that

$$T(1,0,0,0) = a_1,\ T(1,1,0,0) = a_2,\ T(1,1,1,0) = a_3,\ T(1,1,1,1) = a_4.$$

It turns out $\dim(\ker(T)) = 2$. Write $\ker(T)$ as the span of 2 vectors.

Solution: Using part (a), we know that

$$a_1 + a_2 = 0, \quad a_1 - 2a_3 + a_4 = 0.$$

This means that

$$T((1,0,0,0) + (1,1,0,0)) = T(1,0,0,0) + T(1,1,0,0) = a_1 + a_2 = 0$$

and

$$T((1,0,0,0) - 2(1,1,1,0) + (1,1,1,1)) = T(1,0,0,0) - 2T(1,1,1,0) + T(1,1,1,1) = a_1 - 2a_3 + a_4 = 0.$$

This implies $(2,1,0,0)$ and $(0,-1,-1,1)$ are nonparallel vectors in $\ker(T)$. So the kernel is the span of those 2 vectors.

4. Let A and B be equivalent matrices defined by

$$A = \begin{bmatrix} 2 & -1 & 1 & -1 \\ -1 & -1 & 2 & 5 \\ 0 & 2 & 3 & 13 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 3 \end{bmatrix} = B.$$

Define $T : \mathbb{R}^4 \to \mathbb{R}^3$ by $T(x) = Ax$.

(a) (3 points) Write range(T) as a span of some vectors. Is T onto?

> **Solution:** The function T is onto because B has no row of zeros. We have range(T) = span$\{e_1, e_2, e_3\}$.

(b) (3 points) Write null(A) as a span of some vectors. Is T one-to-one?

> **Solution:** The function T is not one-to-one. It is going from a higher dimensional space so a lower one. We have null(A) = span$\{(1, -2, -3, 1)\}$.

(c) (3 points) Let $v = (1, 31, 20, 18)$. Give a vector w different from v such that $T(v) = T(w)$.

> **Solution:** We can add any vector from the kernel of T to v and still have the same output. So $w = v + (1, -2, -3, 1)$ works.

(d) (3 points) Write the first column of A as a linear combination of the second, third, and fourth column of A.

> **Solution:** From the nullspace calculation, we know that $a_1 - 2a_2 - 3a_3 + a_4 = 0$. Rearranging, we derive $a_1 = 2a_2 + 3a_3 - a_4$.

5. Let P be the plane $x + y - z = 0$. Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be the linear transformation given by orthogonal projection onto P. This means

$$T(v) = \begin{cases} 0 & \text{if } v \text{ is normal to } P, \\ v & \text{if } v \text{ is in } P. \end{cases}$$

(a) (3 points) Give a spanning and linearly independent subset of $\mathbb{R}^3$ consisting of a vector normal to P and two vectors that lie in P.

Solution: A normal vector is $a = (1, 1, -1)$. Two linearly independent vectors that lie in P are $b = (1, 0, 1)$ and $c = (0, 1, 1)$.
The set $\{a, b, c\}$ is spanning and linearly independent.

(b) (3 points) There is a matrix A such that $T(x) = Ax$. What is A? You may express A as a product of matrices and their inverses.

Solution: From the description of T, we know that $T(n) = 0$, $T(p) = p$, and $T(q) = q$. So

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ -1 & 1 & 1 \end{bmatrix}^{-1}.$$

(c) (3 points) Is T one-to-one? If not, give two vectors r, s such that $r \neq s$ but $T(r) = T(s)$.

Solution: No. $T(1, 1, -1) = T(0, 0, 0) = (0, 0, 0)$.

(d) (3 points) Is T onto? If not, give a vector in the codomain that is not in the range of T.

Solution: No. $(1, 1, -1)$ is not in the range.