Math 308 Spring '18
Midterm 2
2018-05-18

KEY

Question	Points	Score
1	12	
2	12	
3	12	
4	12	
5	12	
Total:	60	

- There are 5 problems on this exam. Be sure you have all 5 problems on your exam.
- The final answer must be left in exact form. Box your final answer.
- You are allowed the TI-30XIIS calculator. It is possible to complete the exam without a calculator.
- You are allowed a single sheet of 2-sided handwritten self-written notes.
- You must show your work to receive full credit. A correct answer with no supporting work will receive a zero.
- Use the backsides if you need extra space. Make a note of this if you do.
- Do not cheat. This exam should represent your own work. If you are caught cheating, I will report you to the Community Standards and Student Conduct office.

Conventions:

- I will often denote the zero vector by 0.
- When I define a variable, it is defined for that whole question. The A defined in Question 2 is the same for the entire page.
- I often use x to denote the vector $(x_1, x_2, \ldots, x_n)$. It should be clear from context.
- Row and column vectors are distinguished by context. You may ask if you are unsure.

1. If possible, give an example of each of the following. Be sure to explain why your example fits the condition of the problem. If it is not possible, write "NOT POSSIBLE" and briefly explain why. The explanation matters for the grading of this problem. For example, writing "NOT POSSIBLE" with a completely wrong explanation or no explanation could possibly yield **zero** points.

   (a) (3 points) If possible, give an example of a linear transformation $T : \mathbb{R}^5 \to \mathbb{R}^2$ with nullity 1.

   Solution: Not possible. Since the codomain of T is $\mathbb{R}^2$, the rank is at most 2 so the nullity is at least 3.

   (b) (3 points) If possible, give an example of two linear transformations $T : \mathbb{R}^3 \to \mathbb{R}^3$ and $S : \mathbb{R}^3 \to \mathbb{R}^3$ such that $\text{rank}(T) = \text{rank}(S) = 2$ but $\text{rank}(T \circ S) = 1$.

   Solution: Possible. Let $T(x, y, z) = (x, y, 0)$ and $S(x, y, z) = (0, y, z)$.

   (c) (3 points) If possible, give an example of a 2-dimensional subspace of $\mathbb{R}^3$ that contains all of the the vectors $(1, 0, 0), (1, 1, 0), (1, 1, 1)$. You may define your subspace as either a span or a nullspace.

   Solution: Not possible. The given vectors form a basis for $\mathbb{R}^3$ so any subspace containing them must be $\mathbb{R}^3$.

   (d) (3 points) If possible, give an example of a 2×3 matrix A whose columns are not spanning but $Ax = e_1$ has a solution.

   Solution: Possible. Let

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

2. Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be defined by $T(x, y, z) = (x + 2z, x + 2y + z, z)$. There is a matrix A such that $T(x) = Ax$.

   (a) (4 points) What is A?

   Solution:

$$A = \begin{pmatrix} 1 & 0 & 2 \\ 1 & 2 & 1 \\ 0 & 0 & 1 \end{pmatrix}$$

   (b) (4 points) What is A^{-1}?

   Solution:

$$A^{-1} = \begin{pmatrix} 1 & 0 & -2 \\ -\frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ 0 & 0 & 1 \end{pmatrix}$$

   (c) (4 points) Let

$$B = \begin{pmatrix} 5 & 1 & -11 \\ 0 & -1 & 21 \\ 0 & 0 & 1 \end{pmatrix}.$$

   What is $\det(2AB^2)$? You may leave your answer as a product of integers and their exponents. (For example, $2^{100} \cdot 3 \cdot 12$)

   Solution: We have that $\det(A) = 2$, $\det(B) = -5$ so

$$\det(2AB^2) = \det(2I_3)\det(A)\det(B)^2 = 2^3 \cdot 2 \cdot (-5)^2.$$

3. Let A and B be equivalent matrices defined by:

$$A = \begin{pmatrix} 1 & -1 & 3 & -2 \\ 2 & 1 & 0 & 5 \\ 0 & -3 & 6 & -9 \\ 0 & 2 & 1 & 1 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{pmatrix} = B.$$

(a) (3 points) Give a basis for null(A).

Solution: $\{(-2, -1, 1, 1)\}$

(b) (3 points) Give a basis for col(A).

Solution: The first 3 columns of A.

(c) (3 points) Give a basis for row(A).

Solution: The first 3 rows of A.

(d) (3 points) Let a_4 be the fourth column of A. Does there exist a vector $x \in \mathbb{R}^4$ such that $x \in \text{span}\{e_1, e_2, e_3\}$ and $Ax = a_4$? If so, what is an example? If not, why not?

Solution: Yes. Since $a_4 = 2a_1 + a_2 - a_3$, $2e_1 + e_2 - e_3$ is a solution to $Ax = a_4$.

4. Let $\{u_1, u_2, u_3\}$ be a basis with

$$u_1 = (1,2,3), \quad u_2 = (3,0,8), \quad u_3 = (5,18,2018).$$

Let

$$v_1 = (1,-1,2), \quad v_2 = (0,1,-1), \quad v_3 = v_1 + v_2 = (1,0,1).$$

(a) (3 points) Suppose $T : \mathbb{R}^3 \to \mathbb{R}^3$ is a linear transformation defined so that

$$T(e_1) = v_1, \quad T(e_2) = v_2, \quad T(e_3) = v_3.$$

There is a matrix A such that $T(x) = Ax$. What is A?

Solution:

$$A = [v_1\ v_2\ v_3].$$

(b) (3 points) Suppose $S : \mathbb{R}^3 \to \mathbb{R}^3$ is a linear transformation defined so that

$$S(u_1) = v_1, \quad S(u_2) = v_2, \quad S(u_3) = v_3.$$

There is a matrix B so that $S(x) = Bx$. What is B? You may express B as products of explicit matrices and their inverses.

Solution:

$$B = A[u_1\ u_2\ u_3]^{-1}$$

(c) (3 points) Give a basis for range(S).

Solution: We have range(S) = span$\{v_1, v_2, v_3\}$ but v_3 is linearly dependent on v_1, v_2 so a basis is $\{v_1, v_2\}$.

(d) (3 points) Give a basis for ker(S).

Solution: The nullity is 1. A basis is $\{u_1 + u_2 - u_3\}$.

5. Let A and B be equivalent matrices defined by:

$$A = \begin{pmatrix} 1 & 3 & -7 & 1 \\ 2 & 0 & 4 & 0 \\ 3 & 8 & -18 & 1 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & 2 & 0 \\ 0 & 1 & -3 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = B.$$

The usefulness of A and B is for you to decide. Let $a_1, a_2, a_3, a_4 \in \mathbb{R}^3$ be the **columns** of A. Let $b_1, b_2, b_3 \in \mathbb{R}^4$ be the **rows** of B.

Let $S : \mathbb{R}^4 \to \mathbb{R}^3$ be a linear transformation such that $S(b_1) = a_1, S(b_2) = a_2$, and $S(b_3) = a_3$. This problem does not provide complete information about S.

(a) (4 points) Is there enough information to determine $S(2, 1, 1, 1)$? If so, what is it? If not, why not? You may write your answer as a linear combination of a_i's.

Solution: The b_i's form a nice echelon basis for their span so the membership problem is easy here. By staring at it,

$$(2, 1, 1, 1) = 2b_1 + b_2 + b_3.$$

This means

$$S(2, 1, 1, 1) = S(2b_1 + b_2 + b_3) = 2a_1 + a_2 + a_3.$$

(b) (4 points) Is there enough information to determine $S(2, 3, 1, -2)$? If so, what is it? If not, why not? You may write your answer as a linear combination of a_i's.

Solution: Since $(2, 3, 1, -2)$ is not in the span of b_1, b_2, b_3, we cannot determine the value of $S(2, 3, 1, -2)$.

(c) (4 points) Give a nonzero vector in $\ker(S)$. You may write your answer as a linear combination of vectors.

Solution: First observe that $a_3 = 2a_1 - 3a_2$ so $2a_1 - 3a_2 - a_3 = 0$. This means $2b_1 - 3b_2 - b_3 \in \ker(S)$.