Math 308 Spring '18
Midterm 1
2018-04-20

KEY

Question	Points	Score
1	12	
2	12	
3	12	
4	12	
5	12	
Total:	60	

- There are 5 problems on this exam. Be sure you have all 5 problems on your exam.
- The final answer must be left in exact form. Box your final answer.
- You are allowed the TI-30XIIS calculator. It is possible to complete the exam without a calculator.
- You are allowed a single sheet of 2-sided handwritten self-written notes.
- You must show your work to receive full credit. A correct answer with no supporting work will receive a zero.
- Use the backsides if you need extra space. Make a note of this if you do.
- Do not cheat. This exam should represent your own work. If you are caught cheating, I will report you to the Community Standards and Student Conduct office.

Conventions:

- I will often denote the zero vector by 0.
- When I define a variable, it is defined for that whole question. The A defined in Question 2 is the same for the entire page.
- I often use x to denote the vector $(x_1, x_2, \ldots, x_n)$. It should be clear from context.
- Sometimes I write vectors as a row and sometimes as a column. The following are the same to me.

$$(1,2,3) \quad \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}.$$

- I write the evaluation of linear transforms in a few ways. The following are the same to me.

$$T(1,2,3) \quad T((1,2,3)) \quad T\left(\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}\right)$$

1. Give an example of each of the following. If it is not possible, write "NOT POSSIBLE". You do not need to justify.

(a) (3 points) If possible, give an example of a linear system with more variables than equations and only finitely many solutions.

> **Solution:** Possible. Consider the system $x + y + z = 1; x + y + z = 2$. This has no solutions.

(b) (3 points) If possible, give an example of a one-to-one linear transformation $T : \mathbb{R}^3 \to \mathbb{R}^3$ whose range does not contain $(1, 0, 0)$.

> **Solution:** Not possible. Unifying theorem.

(c) (3 points) If possible, give an example of a linear transformation $T : \mathbb{R}^2 \to \mathbb{R}^3$ that is not one-to-one.

> **Solution:** Possible. $T(x) = 0$.

(d) (3 points) If possible, give an example of a linearly dependent set of 3 vectors $\{u, v, w\}$ in $\mathbb{R}^3$ such that $\{u, v\}, \{u, w\}$, and $\{v, w\}$ are linearly independent.

> **Solution:** Possible. Let $u = e_1, v = e_2, w = e_1 + e_2$.

2. Let

$$A = \begin{pmatrix} 1 & 1 & 0 & 3 \\ 0 & 1 & 1 & 1 \\ 1 & 2 & 1 & 4 \end{pmatrix} \quad \text{and} \quad b = \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix}$$

(a) (6 points) Give the general solution to $Ax = b$. (Hint: The sum of the components in the reduced echelon form of A is 5.)

Solution: $x = (3, -1, 0, 0) + s_1(1, -1, 1, 0) + s_2(-2, -1, 0, 1)$.

(b) (3 points) Give 3 particular solutions to $Ax = b$ that do not lie on a line. To save time, you may consider naming some vectors. You may also leave your answer as a linear combination of vectors. (Here we are thinking of vectors as points in space.)

Solution: Many answers. For example, the 3 vectors formed by setting $(s_1, s_2) = (0, 0), (1, 0), (0, 1)$.

(c) (3 points) Does there exists a vector $y \in \mathbb{R}^3$ such that $Ax = y$ has no solutions? Justify your answer. This is a yes-no question.

Solution: Yes. The echelon form of A has a row of zeros so the columns of A are not spanning.

3. Let A and B be equivalent matrices given by

$$A = \begin{pmatrix} 1 & -1 & 3 & 6 \\ 2 & -2 & 0 & 6 \\ 3 & -3 & 8 & 17 \end{pmatrix} \sim \begin{pmatrix} 1 & -1 & 0 & 3 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} = B.$$

Let $S = \{a_1, a_2, a_3, a_4\}$ be the set of columns of A. Let $T : \mathbb{R}^4 \to \mathbb{R}^3$ be a linear transformation defined by $T(x) = Ax$.

The following are quick questions that you should briefly justify with a single sentence or two.

(a) (2 points) Is T one-to-one?

Solution: No. There is a free variable in B.

(b) (2 points) Is T onto?

Solution: No. There is a row of zeros in B.

(c) (2 points) Is S linearly independent?

Solution: No. Same question as (a).

(d) (2 points) Is S spanning?

Solution: No. Same question as (b).

(e) (2 points) Let $b = 4a_1 + 20a_2 + 2018a_3$. What is the dimension of the solution space to $Ax = b$? (Hint: If you claim a solution exists or does not exist, you should explain why.)

Solution: Since b is a linear combination of columns of A, there is a solution. There are 2 free variables so the dimension is 2.

(f) (2 points) Is there a choice of $b \in \mathbb{R}^3$ such that the solution space to $Ax = b$ contains finitely many points? (Same hint as above.)

Solution: Yes. Since the columns of A are not spanning, there exists a b so that $Ax = b$ has no solution.

4. Let

$$u_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, u_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, u_3 = \begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix}.$$

Let $T : \mathbb{R}^3 \to \mathbb{R}^2$ be the linear transformation such that

$$T(u_1) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, T(u_2) = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, T(u_3) = \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

(a) (4 points) Is T one-to-one? Is it onto? Briefly justify.

Solution: The map T is not one-to-one since $T(u_2 + u_3) = T(u_1)$ but $u_2 + u_3 \neq u_1$.
The map T is onto because $T(u_1)$ and $T(u_2)$ are linearly independent vectors contained in the range.

(b) (4 points) Write $e_3 = (0, 0, 1)$ as a linear combination of u_1, u_2, u_3.

Solution: $e_3 = u_3 - 3u_1 - 2u_2$.

(c) (4 points) There is a matrix B such that $T(x) = Bx$. What is B?

Solution: This amounts to determining $T(e_1), T(e_2)$, and $T(e_3)$. We are given $T(u_1), T(u_2)$, and $T(u_3)$.
Since $u_1 = e_1$ and $u_2 = e_2$, we're almost done. Using linearity and the previous part,

$$T(e_3) = T(u_3) - 3T(u_1) - 2T(u_2) = (-2, -1).$$

Since $B = [T(e_1)\ T(e_2)\ T(e_3)]$,

$$B = \begin{pmatrix} 1 & 0 & -2 \\ 0 & 1 & -1 \end{pmatrix}.$$

5. Let A and B be equivalent matrices given by

$$A = \begin{pmatrix} 1 & 3 & 5 & 4 & 13 \\ -1 & 0 & -2 & 2 & -1 \\ 1 & 8 & 10 & 0 & 19 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & 2 & 0 & 3 \\ 0 & 1 & 1 & 0 & 2 \\ 0 & 0 & 0 & 1 & 1 \end{pmatrix} = B$$

As usual, let a_1, a_2, a_3, a_4, a_5 denote the columns of A.

(a) (4 points) Give the homogeneous solution to $Ax = 0$.

(b) (4 points) Give the general solution to $2Ax = a_2 - a_4$. (Hint: $Ae_i = a_i$.)

Solution: Using the hint, we deduce that $(0, 1/2, 0, -1/2, 0)$ is particular solution. The general solution is then

$$x = (0, 1/2, 0, -1/2, 0) + s_1(-2, -1, 1, 0, 0) + s_2(-3, -2, 0, -1, 0).$$

(c) (4 points) The columns corresponding to free variables are a_3 and a_5. Write them as a linear combination of a_1, a_2, a_4.

Solution:

$$a_3 = 2a_1 + a_2$$

$$a_5 = 3a_1 + 2a_2 + a_4$$