1. (10 points) Let $B = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 2 & -1 & 0 & 0 \\ -3 & 3 & 1 & 0 \\ 1 & 2 & -5 & -2 \end{bmatrix}$.

   Calculate the inverse of B, or explain why it does not exist.

   Note that $\det(B) = 1 \cdot -1 \cdot 1 \cdot -2 = 2 \neq 0$ *so B has an inverse.*

   Form the augmented matrix and perform Gauss-Jordan elimination:

$$[B|I] = \left[\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 2 & -1 & 0 & 0 & 0 & 1 & 0 & 0 \\ -3 & 3 & 1 & 0 & 0 & 0 & 1 & 0 \\ 1 & 2 & -5 & -2 & 0 & 0 & 0 & 1 \end{array}\right]$$
$$\sim \left[\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 2 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 & -3 & 3 & 1 & 0 \\ 0 & 0 & 0 & 1 & 10 & -\frac{17}{2} & -\frac{5}{2} & -\frac{1}{2} \end{array}\right]$$

   The inverse matrix is $B^{-1} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 2 & -1 & 0 & 0 \\ -3 & 3 & 1 & 0 \\ 10 & -\frac{17}{2} & -\frac{5}{2} & -\frac{1}{2} \end{bmatrix}$.

2. (10 total points) Determine whether each of the following is a subspace of $\mathbb{R}^4$. Justify your answer.

   (a) (5 points) The set V of vectors of the form $\begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix}$, where $a+b+c=0$ and $b+c+d=0$.

   Yes, V is a subspace of $\mathbb{R}^4$.

   Note that V is the nullspace of the matrix $\begin{bmatrix} 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 \end{bmatrix}$.

   By Theorem 4.3 in the textbook, every nullspace is a subspace.

   (b) (5 points) The set W of vectors of the form $\begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix}$, where $a^2+b^2+c^2+d^2=0$.

   Yes, W is a subspace of $\mathbb{R}^4$.

   The only vector in $\mathbb{R}^4$ that satisfies the condition is the zero vector $\mathbf{0} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$.

   The set $\{\mathbf{0}\}$ *is a subspace of $\mathbb{R}^4$ of dimension zero.*

3. (10 points) Let $A = \begin{bmatrix} -3 & -6 & 6 \\ 3 & 6 & -3 \\ 0 & 0 & 3 \end{bmatrix}$.

   Find the eigenvalues of A. Then find bases for the corresponding eigenspaces of the matrix.

   First, calculate the eigenvalues.

$$\begin{aligned} \det(A-\lambda I) &= \begin{vmatrix} -3-\lambda & -6 & 6 \\ 3 & 6-\lambda & -3 \\ 0 & 0 & 3-\lambda \end{vmatrix} \\ &= (3-\lambda)\cdot[(-3-\lambda)(6-\lambda)+18] \\ &= -\lambda^3+6\lambda^2-9\lambda \\ &= -\lambda(\lambda-3)^2 \end{aligned}$$

   The eigenvalues are $\lambda = 0, 3$. *Now calculate bases for the eigenspaces.*

   $\lambda = 0$*: Form the augmented matrix and perform Gauss-Jordan elimination*

$$[A - 0\cdot I\,|\,\mathbf{0}] = \left[\begin{array}{ccc|c} -3 & -6 & 6 & 0 \\ 3 & 6 & -3 & 0 \\ 0 & 0 & 3 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

   The general solution is $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = x_2 \cdot \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}$, *and a basis for the* 0*-eigenspace is* $\left\{ \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix} \right\}$

   $\lambda = 3$*: Form the augmented matrix and perform Gauss-Jordan elimination*

$$[A - 3\cdot I\,|\,\mathbf{0}] = \left[\begin{array}{ccc|c} -6 & -6 & 6 & 0 \\ 3 & 3 & -3 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

   The general solution is $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = x_2 \cdot \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} + x_3 \cdot \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$,

   and a basis for the 3*-eigenspace is* $\left\{ \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \right\}$

4. (10 points) Find a linear transformation $T : \mathbb{R}^4 \to \mathbb{R}^3$ with kernel $\{\mathbf{0}\}$, or explain why such a linear transformation cannont exist.

   This is not possible.

   Let A be the matrix of T. Then A is 3×4.

   We know $\text{rank}(A) \leq 3$ *because A has 3 rows.*

   But $\text{nullity}(A) + \text{rank}(A) = 4$. *Thus* $\text{nullity}(A) \geq 1$.

   Thus the null space of A is non-trivial.

   The kernel of T equals the null space of A, so it is non-trivial too.

5. (10 points) Find matrices A and B such that $\text{rank}(AB) > \text{rank}(A)$, or explain why such matrices cannot exist.

   This is not possible.

   Assume A is $m \times n$ and B is $n \times \ell$.

   The rank of AB is the dimension of the column space of AB.

   Choose $\mathbf{u} \in \text{col}(AB)$.

   There is a $\mathbf{v} \in \mathbb{R}^\ell$ *such that* $\mathbf{u} = (AB)\mathbf{v}$.

   Write $B\mathbf{v} = \mathbf{w}$. *Then*

$$\begin{aligned} \mathbf{u} &= (AB)\mathbf{v} \\ &= A(B\mathbf{v}) \\ &= A\mathbf{w} \end{aligned}$$

   Thus $\mathbf{u} \in \text{col}(A)$.

   We have shown that $\text{col}(AB) \subseteq \text{col}(A)$. *Thus*

$$\begin{aligned} \text{rank}(AB) &= \dim(\text{col}(AB)) \\ &\leq \dim(\text{col}(A)) \\ &= \text{rank}(A) \end{aligned}$$

   So $\text{rank}(AB)$ *is always less than or equal to* $\text{rank}(A)$.