1. (10 points) Solve the system of linear equations.

$$
\begin{aligned}
-2x_1 + 3x_2 + 7x_3 - 11x_4 &= -3 \\
x_1 \qquad - 2x_3 + \quad x_4 &= 3 \\
x_1 - \quad x_2 - 3x_3 + \ 4x_4 &= 2
\end{aligned}
$$

Give two different particular solutions to the linear system.

Form the augmented matrix and perform Gauss-Jordan elimination:

$$
\begin{aligned}
[A \,|\, \mathbf{b}] &= \left[\begin{array}{rrrr|r} -2 & 3 & 7 & -11 & -3 \\ 1 & 0 & -2 & 1 & 3 \\ 1 & -1 & -3 & 4 & 2 \end{array}\right] \\
&\sim \left[\begin{array}{rrrr|r} 1 & 0 & -2 & 1 & 3 \\ 0 & 1 & 1 & -3 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]
\end{aligned}
$$

This matrix is in reduced echelon form, so x_3 and x_4 are free variables and the general solution is

$$
\begin{aligned}
x_1 &= 2x_3 - x_4 + 3 \\
x_2 &= -x_3 + 3x_4 + 1 \\
x_3 &= x_3 \\
x_4 &= x_4
\end{aligned}
$$

There are many correct choices for the two particular solutions.

For example, taking $x_3 = x_4 = 0$ gives $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \\ 0 \\ 0 \end{bmatrix}$ *and taking $x_3 = 1,\ x_4 = 0$ gives* $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 5 \\ 0 \\ 1 \\ 0 \end{bmatrix}$.

2. (10 points) Find all values of x such that $\text{span}\{\mathbf{v_1}, \mathbf{v_2}, \mathbf{v_3}\} = \text{span}\{\mathbf{v_1}, \mathbf{v_2}\}$ where $\mathbf{v_1} = \begin{bmatrix} 3 \\ 2 \\ -3 \end{bmatrix}$, $\mathbf{v_2} = \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$ and $\mathbf{v_3} = \begin{bmatrix} 2 \\ x \\ -5 \end{bmatrix}$.

By Theorem 2.7, it is enough to have $\mathbf{v_3} \in \text{span}\{\mathbf{v_1}, \mathbf{v_2}\}$.

By Theorem 2.6, this is equivalent to the consistency of the linear system represented by the augmented matrix $[\mathbf{v_2}\ \mathbf{v_1} \mid \mathbf{v_3}]$.

Form the augmented matrix and perform Gaussian elimination:

$$[\mathbf{v_2}\,\mathbf{v_1} \,|\, \mathbf{v_3}] = \left[\begin{array}{cc|c} 1 & 3 & 2 \\ -2 & 2 & x \\ 1 & -3 & -5 \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & 3 & 2 \\ 0 & -6 & -7 \\ 0 & 0 & 3x-16 \end{array}\right]$$

This system is consistent only when $3x - 16 = 0$*, or* $x = 16/3$.

3. (10 points) Determine whether the following set of vectors is linearly independent or linearly dependent. If the set is linearly dependent, express one of the vectors as a linear combination of the others.

$$\mathbf{u_1} = \begin{bmatrix} 1 \\ -1 \\ 1 \\ 1 \end{bmatrix}, \quad \mathbf{u_2} = \begin{bmatrix} -2 \\ 3 \\ -3 \\ -3 \end{bmatrix}, \quad \mathbf{u_3} = \begin{bmatrix} -5 \\ 7 \\ -8 \\ -7 \end{bmatrix}, \quad \text{and} \quad \mathbf{u_4} = \begin{bmatrix} 3 \\ -4 \\ 6 \\ 4 \end{bmatrix}.$$

Form the matrix $A = \begin{bmatrix} 1 & -2 & -5 & 3 \\ -1 & 3 & 7 & -4 \\ 1 & -3 & -8 & 6 \\ 1 & -3 & -7 & 4 \end{bmatrix}$.

The vectors $\mathbf{u_1}, \mathbf{u_2}, \mathbf{u_3}$ *and* $\mathbf{u_4}$ *are linearly independent if and only if the equation* $A\mathbf{x} = \mathbf{0}$ *has only the trivial solution.*

Form the augmented matrix and perform Gauss-Jordan elimination:

$$[A\,|\,\mathbf{0}] = \left[\begin{array}{cccc|c} 1 & -2 & -5 & 3 & 0 \\ -1 & 3 & 7 & -4 & 0 \\ 1 & -3 & -8 & 6 & 0 \\ 1 & -3 & -7 & 4 & 0 \end{array}\right]$$
$$\sim \left[\begin{array}{cccc|c} 1 & 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 3 & 0 \\ 0 & 0 & 1 & -2 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

This matrix is in reduced echelon form, so x_4 *is a free variable. Thus there are many nontrivial solutions and the vectors* $\mathbf{u_1}, \mathbf{u_2}, \mathbf{u_3}$ *and* $\mathbf{u_4}$ *are linearly dependent.*

The general solution is

$$\begin{aligned} x_1 &= x_4 \\ x_2 &= -3x_4 \\ x_3 &= 2x_4 \\ x_4 &= x_4 \end{aligned}$$

Take, for example, $x_4 = 1$ *to get the particular solution* $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 1 \\ -3 \\ 2 \\ 1 \end{bmatrix}$.

This means that $\mathbf{u_1} - 3\mathbf{u_2} + 2\mathbf{u_3} + \mathbf{u_4} = \mathbf{0}$ *so we can easily solve for* $\mathbf{u_1}$.

$\mathbf{u_1} = 3\mathbf{u_2} - 2\mathbf{u_3} - \mathbf{u_4}$

4. (10 points) Find a 3×5 matrix A, in *reduced* echelon form, with free variables x_3 and x_5, such that the general solution of the equation $A\mathbf{x} = \begin{bmatrix} 2 \\ -1 \\ 4 \end{bmatrix}$ is $\mathbf{x} = \begin{bmatrix} 2 \\ -1 \\ 0 \\ 4 \\ 0 \end{bmatrix} + s \begin{bmatrix} 2 \\ -3 \\ 1 \\ 0 \\ 0 \end{bmatrix} + t \begin{bmatrix} 0 \\ 1 \\ 0 \\ 5 \\ 1 \end{bmatrix}$, where s and t are real numbers.

Take $x_3 = s$ and $x_5 = t$. Then the general solution is

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \\ 0 \\ 4 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} 2 \\ -3 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_5 \begin{bmatrix} 0 \\ 1 \\ 0 \\ 5 \\ 1 \end{bmatrix} = \begin{bmatrix} 2x_3 + 2 \\ -3x_3 + x_5 - 1 \\ x_3 \\ 5x_5 + 4 \\ x_5 \end{bmatrix}$$

Working backward, this comes from the augmented matrix

$$[A \,|\, \mathbf{b}] = \left[\begin{array}{ccccc|c} 1 & 0 & -2 & 0 & 0 & 2 \\ 0 & 1 & 3 & 0 & -1 & -1 \\ 0 & 0 & 0 & 1 & -5 & 4 \end{array}\right]$$

Thus

$$A = \begin{bmatrix} 1 & 0 & -2 & 0 & 0 \\ 0 & 1 & 3 & 0 & -1 \\ 0 & 0 & 0 & 1 & -5 \end{bmatrix}$$

5. (10 points) Find a linear transformation $T : \mathbb{R}^2 \to \mathbb{R}^3$ such that
$$T\left(\begin{bmatrix} 3 \\ -2 \end{bmatrix}\right) = \begin{bmatrix} 4 \\ 1 \\ -7 \end{bmatrix}, \quad T\left(\begin{bmatrix} -1 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} -3 \\ 0 \\ 2 \end{bmatrix} \quad \text{and} \quad T\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} -2 \\ 1 \\ 3 \end{bmatrix}$$
or explain why such a linear transformation cannot exist.

Let $\mathbf{v_1} = \begin{bmatrix} 3 \\ -2 \end{bmatrix}$, $\mathbf{v_2} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$ *and* $\mathbf{v_3} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$.

First check if $\mathbf{v_3} \in \operatorname{span}\{\mathbf{v_1}, \mathbf{v_2}\}$.

Form the augmented matrix and perform Gauss-Jordan elimination:

$$[\mathbf{v_2}\,\mathbf{v_1}\,|\,\mathbf{v_3}] = \left[\begin{array}{cc|c} -1 & 3 & 1 \\ 1 & -2 & 0 \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & 0 & 2 \\ 0 & 1 & 1 \end{array}\right]$$

Thus $\mathbf{v_3} = 2\mathbf{v_2} + \mathbf{v_1}$. *(This can also be seen just by inspection.)*

Such a linear transformation cannot exist, because

$$T(2\mathbf{v_2} + \mathbf{v_1}) = T(\mathbf{v_3}) = \begin{bmatrix} -2 \\ 1 \\ 3 \end{bmatrix}$$

but

$$2T(\mathbf{v_2}) + T(\mathbf{v_1}) = 2\begin{bmatrix} -3 \\ 0 \\ 2 \end{bmatrix} + \begin{bmatrix} 4 \\ 1 \\ -7 \end{bmatrix} = \begin{bmatrix} -2 \\ 1 \\ -3 \end{bmatrix}$$