Your Name

Student ID #

- Do not open this exam until you are told to begin. You will have 50 minutes for the exam.
- Check that you have a complete exam. There are 4 questions for a total of 50 points.
- You are allowed to have one single sided, handwritten note sheet. Calculators are not allowed.
- Cheating will result in a zero and be reported to the Dean's Academic Conduct Committee.
- **Show all your work**. With the exception of True/False questions, if there is no work supporting your answer, you will not receive credit for the problem. If you need more space to answer a question, continue on the back of the page, and indicate that you have done so.

Question	Points	Score
1	16	
2	12	
3	11	
4	11	
Total:	50	

1. (16 points) True/False and short answer. No justification is necessary for the True/False questions.

   (a) If A and B are matrices such that $AB = C$, and C is invertible, then A and B are invertible.

   ○ True √ **False**

   (b) If A is a 4×6 matrix, then the maximum value of rank(A) is 6.

   ○ True √ **False**

   (c) If A is an $n \times m$ matrix such that $A^T\mathbf{x} = \mathbf{b}$ is consistent for every $\mathbf{b} \in \mathbb{R}^m$, then row$(A) = \mathbb{R}^m$.

   √ **True** ○ False

   (d) If A is an invertible $n \times n$ matrix, then nullity$(A) = 0$.

   √ **True** ○ False

   (e) If A is a singular matrix that is row equivalent to B, then $\det(A) = \det(B)$.

   √ **True** ○ False

   (f) Give an example of a matrix A such that rank$(A) <$ nullity(A).

   Solution: One example is the zero matrix.

   (g) Find the determinant of the matrix $A = \begin{bmatrix} 1 & 2 & -1 \\ 1 & 1 & 0 \\ -2 & -3 & 0 \end{bmatrix}$.

   Solution: Using an expansion along the third column, we find that

$$\det(A) = -1((1)(-3) - (1)(-2)) = 1.$$

   (h) Is A (the same A as in part (g)) invertible? If so, find A^{-1}. If not, write one column of A as a linear combination of the others.

   Solution: Because $\det(A) \neq 0$, A has an inverse, which we find by reducing

$$\left[\begin{array}{ccc|ccc} 1 & 2 & -1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 1 & 1 \\ -2 & -3 & 0 & 0 & 0 & 1 \end{array}\right] \to \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & 3 & 1 \\ 0 & 1 & 0 & 0 & -2 & -1 \\ 0 & 0 & 1 & -1 & -1 & -1 \end{array}\right]$$

   so

$$A^{-1} = \begin{bmatrix} 0 & 3 & 1 \\ 0 & -2 & -1 \\ -1 & -1 & -1 \end{bmatrix}.$$

2. Let A be the matrix

$$A = \begin{bmatrix} 1 & 0 & 2 & -1 \\ 2 & 1 & 1 & 0 \\ -1 & -1 & 1 & 0 \end{bmatrix}.$$

(a) (4 points) Find a basis for the nullspace of A.

Solution: We can row reduce A to

$$\begin{bmatrix} 1 & 0 & 2 & 0 \\ 0 & 1 & -3 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

which tells us x_3 is a free variable. Writing solutions to $A\mathbf{x} = \mathbf{0}$ in vector form tells us

$$\mathcal{N}(A) = \left\{ s \begin{bmatrix} -2 \\ 3 \\ 1 \\ 0 \end{bmatrix}, s \text{ any real number} \right\}$$

so a basis for the nullspace is

$$\left\{ \begin{bmatrix} -2 \\ 3 \\ 1 \\ 0 \end{bmatrix} \right\}.$$

(b) (4 points) Find a basis for the column space of A.

Solution: Using the work from part (a), a basis for the column space consists of the columns of A corresponding to the columns of the reduced matrix with leading variables, i.e. the first, second and fourth columns. Therefore, a basis for the column space is

$$\left\{ \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 0 \end{bmatrix} \right\}.$$

(c) (4 points) Define a linear transformation $T : \mathbb{R}^4 \to \mathbb{R}^3$ by $T(\mathbf{x}) = A\mathbf{x}$. Is T onto or one-to-one? Justify your answer.

Solution: From part (a), we can see that T has non-trivial kernel ($\ker(T) = \mathcal{N}(A)$) so T is not one-to-one. However, the column space of A is $\mathbb{R}^3$, so the columns of A span $\mathbb{R}^3$, hence T is onto.

3. Let $\mathbf{u}_1 = \begin{bmatrix} 3 \\ 1 \\ 0 \end{bmatrix}$ and $\mathbf{u}_2 = \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix}$, and let $S = \text{span}\{\mathbf{u}_1, \mathbf{u}_2\}$.

   (a) (4 points) Find a matrix A such that the nullspace of A is equal to S.

   Solution: We want to find a matrix A such that $\mathcal{N}(A) = S$, and $\mathcal{N}(A)$ is defined to be the set of all vectors $\mathbf{x}$ such that $A\mathbf{x} = \mathbf{0}$. But, S is the set of all vectors

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = s_1 \begin{bmatrix} 3 \\ 1 \\ 0 \end{bmatrix} + s_2 \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix}$$

   for any real numbers s_1 and s_2, which tells us

$$\begin{aligned} x_1 &= 3s_1 - 2s_2 \\ x_2 &= s_1 \\ x_3 &= s_2. \end{aligned}$$

   In other words, x_2 and x_3 can be any real number and $x_1 = 3x_2 - 2x_3$, or $x_1 - 3x_2 + 2x_3 = 0$, so this is the nullspace of the matrix

$$A = \begin{bmatrix} 1 & -3 & 2 \end{bmatrix}.$$

   (Note that other answers are possible.)

   (b) (4 points) Extend $\{\mathbf{u}_1, \mathbf{u}_2\}$ to a basis for $\mathbb{R}^3$ (i.e., find another vector v such that $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{v}\}$ is a basis for $\mathbb{R}^3$).

   Solution: To extend this to a basis, we just need to find one vector not in S. But, using our work from part (a), any vector $\mathbf{x}$ that is a linear combination of $\mathbf{u}_1$ and $\mathbf{u}_2$ must satisfy $x_1 - 3x_2 + 2x_3 = 0$, so any vector that does not satisfy that equation is not in S. For example, we could choose $\mathbf{v} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ and then $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{v}\}$ is a basis for $\mathbb{R}^3$.

   (c) (3 points) Suppose $T : \mathbb{R}^6 \to \mathbb{R}^3$ is a linear transformation such that the range of T is equal to S. What is the dimension of the kernel of T? Justify your answer.

   Solution: T corresponds to a 3×6 matrix A, and if the range of T is equal to S, the range of T (which is the column space of A) has dimension 2, meaning $\text{rank}(A) = 2$. By the rank-nullity theorem, this means $\text{nullity}(A) = 4$, but the nullity of A is the dimension of the nullspace of A, which is the dimension of the kernel of T. Hence, $\dim(\ker(T)) = 4$.

4. Let $\mathbf{u}_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$ and $\mathbf{u}_2 = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$.

   (a) (2 points) Explain why $\{\mathbf{u}_1, \mathbf{u}_2\}$ is a basis for $\mathbb{R}^2$.

   Solution: These vectors are linearly independent (since they are not multiples of each other) so by the Big Theorem, they span $\mathbb{R}^2$, so they are a basis for $\mathbb{R}^2$.

   (b) (4 points) Write the vector $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ in terms of this basis (i.e., write $\mathbf{x}$ as a linear combination of $\mathbf{u}_1$ and $\mathbf{u}_2$). Your answer should involve x_1 and x_2.

   Solution: Reducing the augmented matrix

$$\left[\begin{array}{cc|c} 1 & 2 & x_1 \\ -1 & -1 & x_2 \end{array}\right]$$

   we get

$$\left[\begin{array}{cc|c} 1 & 0 & -x_1 - 2x_2 \\ 0 & 1 & x_1 + x_2 \end{array}\right]$$

   so

$$\mathbf{x} = (-x_1 - 2x_2)\mathbf{u}_1 + (x_1 + x_2)\mathbf{u}_2.$$

   (c) (5 points) Suppose $T : \mathbb{R}^2 \to \mathbb{R}^2$ is a linear transformation such that $T(\mathbf{u}_1) = \begin{bmatrix} 2 \\ 4 \end{bmatrix}$ and $T(\mathbf{u}_2) = \begin{bmatrix} 3 \\ -1 \end{bmatrix}$. Find a matrix A such that $T(\mathbf{x}) = A\mathbf{x}$. (Hint: use your answer to part (b) to find a formula for $T(\mathbf{x})$).

   Solution: Because T is a linear transformation and $\mathbf{x} = (-x_1 - 2x_2)\mathbf{u}_1 + (x_1 + x_2)\mathbf{u}_2$,

$$\begin{aligned} T(\mathbf{x}) &= T((-x_1 - 2x_2)\mathbf{u}_1 + (x_1 + x_2)\mathbf{u}_2) \\ &= (-x_1 - 2x_2)T(\mathbf{u}_1) + (x_1 + x_2)T(\mathbf{u}_2) \\ &= (-x_1 - 2x_2)\begin{bmatrix} 2 \\ 4 \end{bmatrix} + (x_1 + x_2)\begin{bmatrix} 3 \\ -1 \end{bmatrix} \\ &= \begin{bmatrix} x_1 - x_2 \\ -5x_1 - 9x_2 \end{bmatrix}. \end{aligned}$$

   Because

$$T(\mathbf{x}) = \begin{bmatrix} x_1 - x_2 \\ -5x_1 - 9x_2 \end{bmatrix},$$

   $T(\mathbf{x}) = A\mathbf{x}$ where

$$A = \begin{bmatrix} 1 & -1 \\ -5 & -9 \end{bmatrix}.$$