Math 308 H **Final Exam** **Winter 2017**

Your Name

Student ID #

- Do not open this exam until you are told to begin. You will have 1 hour and 50 minutes for the exam.
- Check that you have a complete exam. There are 6 questions for a total of 80 points.
- You are allowed to have one double sided, handwritten note sheet. Calculators are not allowed.
- Cheating will result in a zero and be reported to the Dean's Academic Conduct Committee.
- **Show all your work**. With the exception of True/False questions, if there is no work supporting your answer, you will not receive credit for the problem. If you need more space to answer a question, continue on the back of the page, and indicate that you have done so.

Question	Points	Score
1	20	
2	8	
3	10	
4	12	
5	12	
6	18	
Total:	80	

1. (20 points) You do not need to show any work for this question.

   (a) For any $n \times n$ matrix A, if $\det A > 0$, then the determinant of each $n-1 \times n-1$ minor of A is also positive.

   ○ True √ **False**

   (b) If $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ is a collection of nonzero orthogonal vectors in $\mathbb{R}^n$, then it is a basis for $\mathbb{R}^n$.

   √ **True** ○ False

   (c) If $T : \mathbb{R}^m \to \mathbb{R}^n$ is an onto linear transformation, and $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k\}$ is a linearly independent set of vectors in $\mathbb{R}^m$, then $\{T(\mathbf{v}_1), T(\mathbf{v}_2), \ldots, T(\mathbf{v}_k)\}$ is linearly independent.

   ○ True √ **False**

   (d) If A and B are $n \times k$ matrices, the set of solutions to the equation $A\mathbf{x} = B\mathbf{x}$ is a subspace of $\mathbb{R}^k$.

   √ **True** ○ False

   (e) If c is an eigenvalue of A, then c^2 is an eigenvalue of A^2.

   √ **True** ○ False

   (f) If span $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ = span $\{\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_k\}$, then $n = k$.

   ○ True √ **False**

   (g) If A and B are matrices such that AB is an $n \times n$ matrix, and $\det(AB) \neq 0$, then A and B are invertible.

   ○ True √ **False**

   (h) If S is a subspace of dimension k and $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ is a set of vectors that spans S, then $n \geq k$.

   √ **True** ○ False

   (i) If A is an $n \times n$ matrix such that $A^T A = I$, then $A = I$.

   ○ True √ **False**

   (j) If A is an $n \times n$ matrix, then row (A) = col (A).

   ○ True √ **False**

2. (8 points) (a) Given 3 data points, $(-1, 0), (0, -1), (2, 1)$, **set up** the linear system to find a line through all three points. You do not need to solve.

Solution: An equation of a line has the form $y = mx + b$. Using the three points, we get a system of equations

$$\begin{aligned} -m + b &= 0 \\ b &= -1 \\ 2m + b &= 1 \end{aligned}$$

which gives us the matrix equation $A\mathbf{x} = \mathbf{y}$ where $A = \begin{bmatrix} -1 & 1 \\ 0 & 1 \\ 2 & 1 \end{bmatrix}$, $\mathbf{x} = \begin{bmatrix} m \\ b \end{bmatrix}$, and $\mathbf{y} = \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}$.

(b) **Set up** the normal equations to find the least-squares solution (the best-fit line through the data points). You do not need to solve.

Solution: The normal equations are $A^T A\mathbf{x} = A^T\mathbf{y}$. Finding A^T and plugging in, we get

$$\begin{bmatrix} 5 & 1 \\ 1 & 3 \end{bmatrix} \mathbf{x} = \begin{bmatrix} 2 \\ 0 \end{bmatrix}$$

(c) Will the normal equations have a unique solution? Explain why or why not. Again, you do not need to solve.

Solution: Because $\det A^T A = 14 \neq 0$, it is invertible, so it will have a unique solution.

3. (10 points) Consider the matrix $A = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix}$.

(a) Find a basis for $W = \text{col}\,(A)$. What is $\dim W$?

Solution: The third column is a linear combination of the first two, and the first two are linearly independent, so a basis for col (A) is

$$\left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix} \right\}$$

and $\dim W = 2$.

(b) Find a basis for $W^\perp$, the orthogonal complement to W. What is $\dim W^\perp$?

Solution: We need to find all vectors orthogonal to $\mathbf{u}_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}$ and $\mathbf{u}_2 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix}$. If $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$, then $\mathbf{x}$ is orthogonal to $\mathbf{u}_1$ and $\mathbf{u}_2$ if and only if $x_1 + x_2 = 0$ and $x_3 + x_4 = 0$. Therefore, a basis $W^\perp$ is

$$\left\{ \begin{bmatrix} 1 \\ -1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ -1 \end{bmatrix} \right\}$$

. Therefore, $\dim W^\perp = 2$.

(c) If $\mathbf{u}$ is any vector in $W^\perp$, what is the closest vector in W to $\mathbf{u}$? Explain your answer.

Solution: Because $\mathbf{u}$ is orthogonal to every vector in W, $\text{proj}_W \mathbf{u} = \mathbf{0}$, so the closest vector to $\mathbf{u}$ is $\mathbf{0}$.

4. (12 points) Let A be the matrix $A = \begin{bmatrix} 1 & -1 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 2 \end{bmatrix}$.

(a) Find the eigenvalues and a basis for each eigenspace of A. Clearly label your answers.

> **Solution:** The characteristic polynomial is $\det(A - \lambda I) = (1-\lambda)^2(2-\lambda)$, so the eigenvalues of A are 1 and 2. To compute the eigenspace of $\lambda = 1$, note that $A - I$ reduces to $\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$, hence the eigenspace of $\lambda = 1$ has basis $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$. To compute the eigenspace of $\lambda = 2$, reduce $A - 2I$ to $\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -2 \\ 0 & 0 & 0 \end{bmatrix}$; hence the eigenspace of $\lambda = 2$ has basis $\begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix}$.

(b) Fill in the following table with all values of λ such that $A - \lambda I$ has the given nullity. If no such λ exists, write DNE.

λ	nullity$(A - \lambda I)$
$\lambda \neq 1, 2$	0
$\lambda = 1, 2$	1
DNE	2
DNE	3

(c) Are there any vectors $\mathbf{x}$ such that the linear transformation $T(\mathbf{x}) = A^T\mathbf{x}$ satisfies $T(\mathbf{x}) = \mathbf{x}$? Explain why or why not.

> **Solution:** Yes. The eigenvalues of A and A^T are the same, so 1 is an eigenvalue of A^T. Therefore, any vector in the eigenspace of $\lambda = 1$ for A^T satisfies $T(\mathbf{x}) = \mathbf{x}$.

5. (12 points) Let A be the matrix $A = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & k \end{bmatrix}$, where k is some real number.

(a) Find all values of k such that A is invertible. Justify your answer.

Solution: Because $\det A = 2k$, A is invertible for every $k \neq 0$.

(b) Find all values of k such that the columns of A do not span $\mathbb{R}^3$. Justify your answer.

Solution: By the Big theorem, the columns do not span $\mathbb{R}^3$ if and only if $\det A = 0$, so the only such value of k is $k = 0$.

(c) Find all values of k such that k is an eigenvalue of A. Justify your answer.

Solution: The characteristic polynomial of A is $\det(A - \lambda I) = (1-\lambda)(2-\lambda)(k-\lambda)$. Therefore, for any value of k, k must be an eigenvalue of A.

(d) Find all values of k such that the linear transformation $T(\mathbf{x}) = A\mathbf{x} - 2\mathbf{x}$ is onto. Justify your answer.

Solution: Because 2 is an eigenvalue of A, the linear transformation $T(\mathbf{x}) = A\mathbf{x} - 2\mathbf{x}$ is never onto. Therefore, there are no values of k such that $T(\mathbf{x}) = A\mathbf{x} - 2\mathbf{x}$ is onto.

6. (18 points) Give an example of each of the following or explain why one cannot exist.

(a) A 3×4 matrix A such that col (A) is the plane $x + y + z = 0$.

Solution: One example is the matrix $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & -1 & 0 & 0 \end{bmatrix}$.

(b) A nonzero linear transformation $T : \mathbb{R}^2 \to \mathbb{R}^2$ and a nonzero vector $\mathbf{v}$ such that $T(\mathbf{x} + \mathbf{v}) = T(\mathbf{x})$ for all $\mathbf{x}$ in $\mathbb{R}^2$.

Solution: One example is $T(\mathbf{x}) = A\mathbf{x}$ where A is the matrix $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$.

(c) A basis $\mathcal{B}$ such that $\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}_{\mathcal{B}}$ and $\begin{bmatrix} 4 \\ 0 \end{bmatrix} = \begin{bmatrix} -1 \\ -1 \end{bmatrix}_{\mathcal{B}}$.

Solution: The only possible example is the basis $\left\{ \begin{bmatrix} -9 \\ -1 \end{bmatrix}, \begin{bmatrix} 5 \\ 1 \end{bmatrix} \right\}$.

(d) A square matrix A other than the identity matrix such that $A^2 = I$ but -1 is not an eigenvalue of A.

Solution: Such a matrix cannot exist. If $A^2 = I$, then $A^2 - I = 0$, meaning $(A - I)(A + I) = 0$. Because -1 is not an eigenvalue of A, $A + I$ has an inverse, so we can multiply both sides by $(A + I)^{-1}$ to get that $A - I = 0$, or $A = I$.

(e) Matrices A and B such that $AB = I$ where B has more columns than rows.

Solution: Such matrices cannot exist. If B has more columns than rows, then its columns must be linearly dependent. Therefore, the columns of AB are linearly dependent so AB cannot equal I.

(f) A singular 3×3 matrix A with eigenvalues $1, 2$, and 3.

Solution: This cannot exist. If A is singular, then 0 is an eigenvalue of A, so A would have to have at least 4 eigenvalues, meaning it must be at least a 4×4 matrix.