Name:______________________________ Signature:______________________________

1. (30 points) In the following, each correct answer is worth 2 points. There is no penalty for incorrect answers. You do not need to justify your answers.

   (a) The rank of a matrix is

   - ☐ the dimension of its range.
   - ☐ the dimension of its null space.
   - ☐ both of the above
   - ☐ neither of the above

   (b) Write down an *orthogonal* basis for span$\left\{ \begin{pmatrix} 7 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 5 \\ 2 \\ 0 \end{pmatrix}, \begin{pmatrix} 2 \\ -3 \\ 0 \end{pmatrix}, \begin{pmatrix} 5 \\ 4 \\ 0 \end{pmatrix}, \begin{pmatrix} -7 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \right\}$.

   (c) Let A be a 3×3 matrix whose only eigenvalue is 4, with associated eigenspace all of $\mathbb{R}^3$. Find A.

   (d) Let $A = \begin{pmatrix} \mathbf{a}_1 & \mathbf{a}_2 & \mathbf{a}_3 \end{pmatrix}$ be a square matrix with columns $\mathbf{a}_1$, $\mathbf{a}_2$ and $\mathbf{a}_3$, where $\mathbf{a}_1 = \mathbf{a}_2 + \mathbf{a}_3$. Find $\det(A)$.

   (e) Let A be a 3×3 matrix with $\text{rank}(A) = 3$. What is $\text{rank}(A^{-1})$?

   (f) If $A = \begin{pmatrix} 2 & 0 & 5 & 1 \\ 0 & 3 & 2 & 0 \\ 0 & 0 & 4 & 1 \\ 0 & 0 & 0 & 7 \end{pmatrix}$, list all the eigenvalues of A.

   (g) Give an example of a matrix whose domain is $\mathbb{R}^3$ and range is span$\left\{ \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\}$.

(h) Find a vector $\mathbf{v}$ so that $\left\{ \begin{pmatrix} 1 \\ -1 \end{pmatrix}, \mathbf{v} \right\}$ is an *orthogonal* basis for $\mathbb{R}^2$.

(i) Give an example of a nonzero vector $\mathbf{v}$ that lies in $S^\perp$, if $S = \operatorname{span}\left\{ \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right\}$.

(j) If $S = \operatorname{span}\left\{ \begin{pmatrix} 0 \\ 2 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 17 \\ -8 \end{pmatrix} \right\}$, find $\operatorname{proj}_S \begin{pmatrix} 1 \\ 3 \\ -1 \end{pmatrix}$.

(k) Write down a basis for the null space of $\begin{pmatrix} 2 & 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 & 0 \end{pmatrix}$.

(l) If $A = \begin{pmatrix} 1/2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1/3 \end{pmatrix}$, what is A^{-1}?

(m) Let $A = \begin{pmatrix} 2 & 0 \\ 0 & 1/2 \end{pmatrix}$. For which vectors $\mathbf{x}$ does $\lim_{k\to\infty} A^k\mathbf{x}$ exist?

(n) Let A be a 2×2 matrix with eigenvalues 0 and 4. What is the rank of A?

(o) If A is a noninvertible square matrix, then the system $A\mathbf{x} = \mathbf{0}$ has

- ☐ no solution.
- ☐ a unique solution.
- ☐ infinitely many solutions.

2. (7 points) Let $A = \begin{pmatrix} 4 & 2 \\ -3 & -1 \end{pmatrix}$. Find all eigenvalues of A and their associated eigenspaces.

3. (3 points) Compute $\det(A)$, if $A = \begin{pmatrix} 1 & 3 & 0 \\ -1 & 5 & 4 \\ 0 & 2 & 1 \end{pmatrix}$.

4. (5 points) Let $S = \text{span}\left\{ \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} \right\}$. Compute $\text{proj}_S \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$.

5. (5 points) If A is a matrix such that $A\begin{pmatrix} 2 \\ -1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ and $A\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, find A.

6. (10 points) Let $A = \begin{pmatrix} 2 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}$.

   (a) (5 points) Find a basis for Range(A).

   (b) (5 points) Find a basis for Null(A).

7. (10 points) Let $S = \text{span}\left\{ \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix}, \begin{pmatrix} 2 \\ 4 \\ 0 \end{pmatrix}, \begin{pmatrix} 3 \\ -1 \\ -1 \end{pmatrix} \right\}$.

   (a) (4 points) What is $\dim(S^{\perp})$? Explain either in 1-2 sentences or by drawing a picture.

   (b) (6 points) Find an *orthogonal* basis for S. It may help to recall that $(\mathbf{u} - \text{proj}_{\mathbf{v}}\mathbf{u}) \cdot \mathbf{v} = 0$, for any nonzero vectors $\mathbf{u}$ and $\mathbf{v}$.

8. (10 points) Find a matrix A such that

   - $\text{Null}(A) = \text{span}\left\{ \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} \right\}$,
   - $\text{Range}(A) = (\text{Null}(A))^{\perp}$, and
   - 6 is an eigenvalue of A.