NAME Solutions

Math 308E
Spring 2016

Midterm 1
April 20, 2016

Instructions

- Point totals for each problem are shown in parentheses.
- You must show all your work on the examination to receive credit. You must also use the techniques of this course on each problem; if you have taken a course in linear algebra previously, you may not use anything from that course which was not covered here. Ask if you are unsure of what is allowed.
- Read each problem carefully. You will not receive credit if you misunderstand or misread a problem.
- Your work must be neat and organized.
- Be very careful with your arithmetic. None of the calculations or answers are too complicated.
- Make sure your test has 5 questions.

(6) 1. Find all solutions to the system of linear equations

$$\begin{aligned} 6x_1 + 4x_2 + 3x_3 - 3x_4 &= 12 \\ 2x_1 + x_2 \qquad - x_4 &= 4 \\ -2x_1 \qquad + 4x_3 - x_4 &= -1 \end{aligned}$$

$$\left[\begin{array}{cccc|c} 6 & 4 & 3 & -3 & 12 \\ 2 & 1 & 0 & -1 & 4 \\ -2 & 0 & 4 & -1 & -1 \end{array}\right] \to \left[\begin{array}{cccc|c} 2 & 1 & 0 & -1 & 4 \\ 6 & 4 & 3 & -3 & 12 \\ -2 & 0 & 4 & -1 & -1 \end{array}\right]$$

$$\to \left[\begin{array}{cccc|c} 2 & 1 & 0 & -1 & 4 \\ 0 & 1 & 3 & 0 & 0 \\ 0 & 1 & 4 & -2 & 3 \end{array}\right] \to \left[\begin{array}{cccc|c} 2 & 1 & 0 & -1 & 4 \\ 0 & 1 & 3 & 0 & 0 \\ 0 & 0 & 1 & -2 & 3 \end{array}\right]$$

free variable: x_4.

general solution: $x_4 = s$

$x_3 = 3 + 2s$

$x_2 = -3x_3 = -9 - 6s$

$x_1 = \frac{1}{2}(4 + x_4 - x_2) = \frac{13}{2} + \frac{7s}{2}$

(s any real number)

(6) 2. Find all values c for which the equation

$$\begin{bmatrix} 2 & 4 & -1 \\ 1 & 5 & 1 \\ 0 & 2 & 1 \\ 1 & 4 & c \end{bmatrix} \mathbf{x} = \begin{bmatrix} 0 \\ 3 \\ 2 \\ 2 \end{bmatrix}$$

has no solutions, exactly one solution, and an infinite number of solutions.

$$\left[\begin{array}{ccc|c} 2 & 4 & -1 & 0 \\ 1 & 5 & 1 & 3 \\ 0 & 2 & 1 & 2 \\ 1 & 4 & c & 2 \end{array}\right] \longrightarrow \left[\begin{array}{ccc|c} 1 & 5 & 1 & 3 \\ 2 & 4 & -1 & 0 \\ 0 & 2 & 1 & 2 \\ 1 & 4 & c & 2 \end{array}\right] \longrightarrow \left[\begin{array}{ccc|c} 1 & 5 & 1 & 3 \\ 0 & -6 & -3 & -6 \\ 0 & 2 & 1 & 2 \\ 0 & -1 & c-1 & -1 \end{array}\right]$$

$$\longrightarrow \left[\begin{array}{ccc|c} 1 & 5 & 1 & 3 \\ 0 & 2 & 1 & 2 \\ 0 & 2 & 1 & 2 \\ 0 & -1 & c-1 & -1 \end{array}\right]$$

$$\longrightarrow \left[\begin{array}{ccc|c} 1 & 5 & 1 & 3 \\ 0 & 2 & 1 & 2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & (c-1)+\frac{1}{2} & 0 \end{array}\right]$$

exactly one solution if $(c-1)+\frac{1}{2} \neq 0$; i.e. $c \neq \frac{1}{2}$

infinitely many solutions if $c = \frac{1}{2}$

(6) 3. Let $\mathbf{v}_1 = (1,2,1)$, $\mathbf{v}_2 = (3,-1,1)$, and $\mathbf{v}_3 = (-1,5,1)$ be vectors in $\mathbf{R}^3$. Find a vector not in the span of $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$. (You must show how you know that the vector you find is not in the span; just writing down a vector is not sufficient.)

The vector $\vec{b} = (b_1, b_2, b_3)$ is in the span of $\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ if and only if the equation

$$\begin{bmatrix} \vec{v}_1 & \vec{v}_2 & \vec{v}_3 \end{bmatrix} \vec{x} = \vec{b}$$ has a solution

$$\left[\begin{array}{ccc|c} 1 & 3 & -1 & b_1 \\ 2 & -1 & 5 & b_2 \\ 1 & 1 & 1 & b_3 \end{array}\right] \longrightarrow \left[\begin{array}{ccc|c} 1 & 3 & -1 & b_1 \\ 0 & -7 & 7 & b_2 - 2b_1 \\ 0 & -2 & 2 & b_3 - b_1 \end{array}\right]$$

$$\longrightarrow \left[\begin{array}{ccc|c} 1 & 3 & -1 & b_1 \\ 0 & -7 & 7 & b_2 - 2b_1 \\ 0 & 0 & 0 & (b_3 - b_1) - \frac{2}{7}(b_2 - 2b_1) \end{array}\right]$$

This system has a solution if and only if $b_3 - b_1 - \frac{2}{7}(b_2 - 2b_1) = 0$. So for example there is no solution if $(b_1, b_2, b_3) = (0,0,1)$, and hence $(0,0,1)$ is not in the span of $\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$.

(4) 4. Find a 3×4 matrix A, in *reduced* echelon form, with free variable x_3, such that the general solution of the equation $A\mathbf{x} = \begin{bmatrix} -1 \\ 1 \\ 6 \end{bmatrix}$ is

$$\mathbf{x} = \begin{bmatrix} -1 \\ 1 \\ 0 \\ 6 \end{bmatrix} + s \begin{bmatrix} -1 \\ 2 \\ 1 \\ 0 \end{bmatrix},$$

where s is any real number.

We must have $A = \begin{bmatrix} 1 & 0 & a & 0 \\ 0 & 1 & b & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$ with the first two entries in the 3rd column to be determined. But $\vec{x} = \begin{bmatrix} -2 \\ 3 \\ 1 \\ 6 \end{bmatrix}$ is a solution, this implies that $-2 + a = -1$ and $3 + b = 1$. Hence $a = 1$, $b = -2$, and $A = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & -2 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$.

(3) 5. Find vectors $\mathbf{u}_1$, $\mathbf{u}_2$, and $\mathbf{u}_3$ in $\mathbf{R}^3$ such that $\{\mathbf{u}_1, \mathbf{u}_2\}$, $\{\mathbf{u}_1, \mathbf{u}_3\}$, and $\{\mathbf{u}_2, \mathbf{u}_3\}$ are all linearly independent but $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$ isn't, or explain why such vectors cannot exist.

Take $\vec{u}_1 = (1,0,0)$, $\vec{u}_2 = (0,1,0)$, $\vec{u}_3 = (1,1,0)$