NAME Solutions

Math 308E
Spring 2016

Final
June 8, 2016

Instructions

- Point totals for each problem are shown in parentheses.
- You must show all your work on the examination to receive credit (except for the first problem). You must also use the techniques of this course on each problem. Ask if you are not sure about what is permitted.
- Read each problem carefully. You will not receive credit if you misunderstand or misread a problem.
- Your work must be neat and organized.
- Be very careful with your arithmetic. None of the calculations or answers are too complicated.
- Make sure your test has 6 questions.

(16) 1. Determine whether each of the following statements is true or false. Put T in the box if true and F if false. No explanation is required. (2 points for each correct answer, 0 points if no answer, -2 points if incorrect answer. The minimum for the entire problem is 0.)

$\boxed{T}$ a. If S is a subspace of $\mathbf{R}^n$ and $\dim S = n$, then $S = \mathbf{R}^n$.

$\boxed{F}$ b. If $A = \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix}$ and B is a 2×2 matrix with $AB = 0$, then B is the 0 matrix.

$\boxed{T}$ c. If A is a 3×4 matrix with $\operatorname{rank} A = 3$, then there is a 4×3 matrix B with $AB = I_3$.

$\boxed{T}$ d. If A is an invertible $n \times n$ matrix and B is an $n \times n$ matrix, then $\operatorname{rank}(AB) = \operatorname{rank}(BA)$.

$\boxed{F}$ e. If A is a 3×4 matrix, then $\operatorname{nullity}(A) = \operatorname{nullity}(A^T)$.

$\boxed{F}$ f. If B is obtained from A through elementary row operations, then the column space of A is equal to the column space of B.

$\boxed{F}$ g. If A is an invertible matrix and $\det(A) > 1$, then 1 is not an eigenvalue of A.

$\boxed{T}$ h. If A is an invertible matrix and $A^2 - A$ is not invertible, then 1 is an eigenvalue of A.

(5) 2. Determine whether span$\{(-1,4,2),(2,3,1)\}$ = span$\{(1,7,3),(-4,5,3)\}$. You must justify your answer to get any credit.

$$(1,7,3) = (-1,4,2) + (2,3,1)$$

$$\left[\begin{array}{cc|c} -1 & 2 & -4 \\ 4 & 3 & 5 \\ 2 & 1 & 3 \end{array}\right] \longrightarrow \left[\begin{array}{cc|c} -1 & 2 & -4 \\ 0 & 11 & -11 \\ 0 & 5 & -5 \end{array}\right] \longrightarrow \left[\begin{array}{cc|c} -1 & 2 & -4 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{array}\right]$$

Therefore,

$$(-4,5,3) = 2(-1,4,2) - (2,3,1),$$

and span$\{(1,7,3),(-4,5,3)\}$ is contained in span$\{(-1,4,2),(2,3,1)\}$. Since both spanning sets are linearly independent, both subspaces have dimension 2 and are therefore equal.

(5) 3. Find the eigenvalues and bases for the corresponding eigenspaces of the matrix

$$A = \begin{bmatrix} 2 & 1 & 2 \\ 0 & 2 & 1 \\ 0 & -1 & 0 \end{bmatrix}.$$

$$\det(A - \lambda I_3) = \begin{vmatrix} 2-\lambda & 1 & 2 \\ 0 & 2-\lambda & 1 \\ 0 & -1 & -\lambda \end{vmatrix}$$

$$= (2-\lambda)\left[(2-\lambda)(-\lambda) + 1\right]$$

$$= (2-\lambda)\left[\lambda^2 - 2\lambda + 1\right] = (2-\lambda)(\lambda-1)^2.$$

eigenvalues: 1, 2.

Eigenspace for $\lambda = 1$ = null$(A - I_3)$

$$\begin{bmatrix} 1 & 1 & 2 \\ 0 & 1 & 1 \\ 0 & -1 & -1 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 1 & 2 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}.$$

free variable x_3

General solution

$x_1 = -s$

$x_2 = -s$

$x_3 = s$

$$\vec{x} = s\begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix}$$

s any number

basis = $\left\{\begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix}\right\}$

Eigenspace for $\lambda = 2$ = null$(A - 2I_3)$

$$\begin{bmatrix} 0 & 1 & 2 \\ 0 & 0 & 1 \\ 0 & -1 & -2 \end{bmatrix} \longrightarrow \begin{bmatrix} 0 & 1 & 2 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

free variable x_1

General solution

$x_1 = s$

$x_2 = 0$

$x_3 = 0$

$$\vec{x} = s\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

s any number

basis = $\left\{\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}\right\}$

(5) 4. Suppose that S is a subspace of $\mathbf{R}^3$ and that $\text{proj}_S(\mathbf{x})$ is given by

$$\text{proj}_S(\mathbf{x}) = \frac{1}{3}\begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix}\mathbf{x}.$$

Find an orthogonal basis for S.

$\text{proj}_S : \mathbb{R}^3 \longrightarrow \mathbb{R}^3$ is a linear transformation whose range is S. ($\text{proj}_S(\vec{x})$ is a vector in S for every $\vec{x}$ in $\mathbb{R}^3$, and, if $\vec{x}$ is in S, $\text{proj}_S(\vec{x}) = \vec{x}$.)

Therefore,

$$S = \text{col}\begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix} = \text{span}\left\{\begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}\right\}.$$

Since

$$\begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix} - \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix},$$

we have

$$S = \text{span}\left\{\begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}\right\}.$$

$\left\{\begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}\right\}$ are linearly independent, so form a basis for S. Now use the Gram-Schmidt orthogonalization process to get orthogonal basis $\{\vec{v}_1, \vec{v}_2\}$, where

$$\vec{v}_1 = \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}$$

$$\vec{v}_2 = \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix} - \frac{\begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix} \cdot \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}}{6}\begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix} - \frac{1}{2}\begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 3/2 \\ -\frac{3}{2} \end{bmatrix}.$$

(5) 5. Let S be the subspace of $\mathbf{R}^4$ spanned by $(-1, 1, 2, 4)$ and $(1, -1, 0, -1)$. Find a basis for $S^{\perp}$.

$$S^{\perp} = \text{null} \begin{bmatrix} -1 & 1 & 2 & 4 \\ 1 & -1 & 0 & -1 \end{bmatrix}$$

$$\begin{bmatrix} -1 & 1 & 2 & 4 \\ 1 & -1 & 0 & -1 \end{bmatrix} \longrightarrow \begin{bmatrix} -1 & 1 & 2 & 4 \\ 0 & 0 & 2 & 3 \end{bmatrix}$$

free variables x_2, x_4

general solution:

$$x_1 = s_1 + s_2$$
$$x_2 = s_1$$
$$x_3 = -\tfrac{3}{2} s_2$$
$$x_4 = s_2$$

$$\vec{x} = s_1 \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix} + s_2 \begin{bmatrix} 1 \\ 0 \\ -3/2 \\ 1 \end{bmatrix}$$

s_1, s_2 any numbers

basis for $S^{\perp}$: $\left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ -3/2 \\ 1 \end{bmatrix} \right\}$

(5) 6. Find the least squares solution to

$$\begin{bmatrix} 1 & 1 \\ 3 & -2 \\ 0 & 1 \end{bmatrix} \mathbf{x} = \begin{bmatrix} -1 \\ 1 \\ -2 \end{bmatrix}.$$

Let $A = \begin{bmatrix} 1 & 1 \\ 3 & -2 \\ 0 & 1 \end{bmatrix}$, $\vec{y} = \begin{bmatrix} -1 \\ 1 \\ -2 \end{bmatrix}$.

$$\hat{x} = (A^T A)^{-1} A^T \vec{y}$$

$$A^T \vec{y} = \begin{bmatrix} 1 & 3 & 0 \\ 1 & -2 & 1 \end{bmatrix} \begin{bmatrix} -1 \\ 1 \\ -2 \end{bmatrix} = \begin{bmatrix} 2 \\ -5 \end{bmatrix}$$

$$A^T A = \begin{bmatrix} 1 & 3 & 0 \\ 1 & -2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 3 & -2 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 10 & -5 \\ -5 & 6 \end{bmatrix}$$

$$(A^T A)^{-1} = \frac{1}{35} \begin{bmatrix} 6 & 5 \\ 5 & 10 \end{bmatrix}$$

$$\hat{x} = \frac{1}{35} \begin{bmatrix} 6 & 5 \\ 5 & 10 \end{bmatrix} \begin{bmatrix} 2 \\ -5 \end{bmatrix} = \frac{1}{35} \begin{bmatrix} -13 \\ -40 \end{bmatrix}$$