MATH 308 M

Exam II

February 21, 2020

Name ____________________________________

Student ID #____________________

HONOR STATEMENT

"I affirm that my work upholds the highest standards of honesty and academic integrity at the University of Washington, and that I have neither given nor received any unauthorized assistance on this exam."

SIGNATURE:________________________________

1	16	
2	6	
3	14	
4	14	
Bonus	~~8~~ 8 OL	
Total	50	

SOLUTIONS!

- Your exam should consist of this cover sheet, followed by 4 problems and a bonus question. Check that you have a complete exam.
- Pace yourself. You have ~~60~~ 50 minutes to complete the exam and there are 4 problems. Try not to spend more than ~~12~~ 10 minutes on each problem. ✓
- Show all your work and justify your answers.
- Your answers should be exact values rather than decimal approximations. (For example, $\frac{\pi}{4}$ is an exact answer and is preferable to its decimal approximation 0.7854.)
- You may use TI-30X IIS calculator and one 8.5×11-inch sheet of handwritten notes. All other electronic devices (including graphing calculators) are forbidden.
- The use of headphones or earbuds during the exam is not permitted.
- There are multiple versions of the exam, you have signed an honor statement, and cheating is a hassle for everyone involved. DO NOT CHEAT.
- Turn your cell phone OFF and put it AWAY for the duration of the exam.

GOOD LUCK!

1. (16 Points) True / False and Short Answer.

   Clearly indicate whether the statement is true or false and **justify your answer**.

   (a) **TRUE** / **FALSE** $\mathbb{R}^2$ is a subspace of $\mathbb{R}^4$.

   $\mathbb{R}^2$ is not a subset of $\mathbb{R}^4$.

   (b) **TRUE** / **FALSE** Let A be an $n \times m$ matrix such that $A^T\vec{x} = \vec{b}$ is a consistent linear system for every $\vec{b}$ in $\mathbb{R}^m$. Then $n < m$.

   A^T is an $m \times n$ matrix, and the statement "$A^T\vec{x} = \vec{b}$ is a consistent linear system for every $\vec{b}$ in $\mathbb{R}^m$" just means $A^T\vec{x} = \vec{b}$ has a solution for any $\vec{b}$ in $\mathbb{R}^m$... ie.

   $$T(\vec{x}) = A^T\vec{b}, \quad T: \mathbb{R}^n \longrightarrow \mathbb{R}^m \quad (A^T \; m \times n)$$

   is onto. This is only possible if $n \geq m$, so the answer is false.

Give an example of each of the following. If there is no such example, write NOT POSSIBLE and **justify why it is not possible.** If you provide an example, you do not need to justify why the example works.

(c) **Give an example** of a matrix A such that $A^8 = I_2$, but $A^k \neq I_2$ for an integer k such that $0 < k < 8$.

rotate by $\frac{\pi}{4}$: $\begin{bmatrix} \cos(\frac{\pi}{4}) & -\sin(\frac{\pi}{4}) \\ \sin(\frac{\pi}{4}) & \cos(\frac{\pi}{4}) \end{bmatrix} = \begin{bmatrix} \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \end{bmatrix}$.

(d) **Give an example** of a linear transformation $T : \mathbb{R}^3 \mapsto \mathbb{R}^3$ such that Range(T) = Ker(T). [*Hint:* Consider the Rank-Nullity Theorem.]

NOT POSSIBLE. By rank-nullity,

$$\underbrace{\dim(\text{range}(T))}_{\text{rank}} + \underbrace{\dim(\ker(T))}_{\text{nullity}} = \underbrace{3}_{\text{dim of domain}}$$

If Range(T) = Ker(T), then they have the same dimension, so their sum will always be even !

(e) **Give an example** of a linear transformation $T : \mathbb{R}^m \mapsto \mathbb{R}^n$ where $m < n$ and T is not one-to-one.

Let $m = 2$, $n = 3$

Let $A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$ 3×2, define $T(\vec{x}) = A\vec{x} = \begin{bmatrix} x_1 \\ 0 \\ 0 \end{bmatrix}$.

f.v.

2. (6 points) ~~Short Answer. Fully justify your reasoning~~.

   (a) Recall from M126 that the Taylor series for the exponential function is $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$. Since we know how to take powers of a matrix, we can use this to define the exponential of a square $n \times n$ matrix A:

$$e^A = \sum_{n=0}^{\infty} \frac{A^n}{n!} \quad \left(\frac{1}{n!}\right)A$$

and $0! = 1$

where we use the convention that $A^0 = I_n$. Compute e^A where A is the matrix $\begin{bmatrix} 2 & -1 \\ 4 & -2 \end{bmatrix}$.

Hint: If you compute the powers of the matrix correctly, you will not need to take an infinite sum.

$$A^0 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$A^1 = \begin{bmatrix} 2 & -1 \\ 4 & -2 \end{bmatrix}$$

$$A^2 = \begin{bmatrix} 2 & -1 \\ 4 & -2 \end{bmatrix}\begin{bmatrix} 2 & -1 \\ 4 & -2 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \quad A^3 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \cdots \quad A^n = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \quad n \geq 2$$

Then

$$e^A = \sum_{n=0}^{\infty} \frac{A^n}{n!} = \frac{1}{0!}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \frac{1}{1!}\begin{bmatrix} 2 & -1 \\ 4 & -2 \end{bmatrix} + \frac{1}{2!}\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} + \cdots = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} 2 & -1 \\ 4 & -2 \end{bmatrix}$$

$$= \begin{bmatrix} 3 & -1 \\ 4 & -1 \end{bmatrix}.$$

   (b) (**BONUS:** ~~2~~ 3 **points**) Using the same instructions from part (a), compute e^A where A is the matrix $\begin{bmatrix} 7 & 0 \\ 0 & 9 \end{bmatrix}$.

$A = \begin{bmatrix} 7 & 0 \\ 0 & 9 \end{bmatrix}$, diagonal, so $A^n = \begin{bmatrix} 7^n & 0 \\ 0 & 9^n \end{bmatrix}$. Then

$$e^A = \sum_{n=0}^{\infty} \frac{1}{n!}\begin{bmatrix} 7^n & 0 \\ 0 & 9^n \end{bmatrix} = \begin{bmatrix} \sum_{n=0}^{\infty} \frac{7^n}{n!} & 0 \\ 0 & \sum_{n=0}^{\infty} \frac{9^n}{n!} \end{bmatrix} = \begin{bmatrix} e^7 & 0 \\ 0 & e^9 \end{bmatrix}.$$

3. (14 Points) Let T be a linear transformation such that $T(\vec{x}) = A\vec{x}$ where A and its equivalent reduced row echelon form is given by

$$A = \begin{bmatrix} 0 & 2 & 6 & 14 \\ 2 & -1 & 1 & 3 \\ 1 & -1 & -1 & 2 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 2 & 9 \\ 0 & 1 & 3 & 7 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

(a) What is the domain of T?

A is 3×4, so $T: \mathbb{R}^4 \to \mathbb{R}^3$. Domain is $\mathbb{R}^4$.

(b) What is the codomain of T?

$\mathbb{R}^3$

(c) Is the null space of A in the domain or codomain? Give a basis for Null(A).

Domain!

$$\left[\begin{array}{cccc|c} 0 & 2 & 6 & 14 & 0 \\ 2 & -1 & 1 & 3 & 0 \\ 1 & -1 & -1 & 2 & 0 \end{array}\right] \sim \left[\begin{array}{cccc|c} 1 & 0 & 2 & 9 & 0 \\ 0 & 1 & 3 & 7 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

f.v.

$x_4 = t$

$x_3 = s$

$x_2 = -3s - 7t$

$x_1 = -2s - 9t$

$$\Rightarrow \vec{x} = \begin{bmatrix} -2s-9t \\ -3s-7t \\ s \\ t \end{bmatrix}$$

so $\vec{x} = s\begin{bmatrix} -2 \\ -3 \\ 1 \\ 0 \end{bmatrix} + t\begin{bmatrix} -9 \\ -7 \\ 0 \\ 1 \end{bmatrix}$, null$(A) = \text{span}\left\{\begin{bmatrix} -2 \\ -3 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -9 \\ -7 \\ 0 \\ 1 \end{bmatrix}\right\}$, and

since these two vectors are necessarily lin. ind, we have $\mathcal{B}_{\text{null}(A)} = \left\{\begin{bmatrix} -2 \\ -3 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -9 \\ -7 \\ 0 \\ 1 \end{bmatrix}\right\}$

(d) Is the column space of A in the domain or codomain? Give a basis for Col(A).

(Method 2) $\mathcal{B}_{\text{col}(A)} = \left\{\begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ -1 \\ -1 \end{bmatrix}\right\}$

(e) Is the row space of A in the domain or codomain? Give a basis for Row(A).

(Method 1) $\mathcal{B}_{\text{row}(A)} = \left\{\begin{bmatrix} 1 \\ 0 \\ 2 \\ 9 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 3 \\ 7 \end{bmatrix}\right\}$

(f) Is T one-to-one? Onto? **Justify your answer.**

Not one-to-one since null$(A) \neq \{\vec{0}\}$, i.e. Ker$(T) \neq \{\vec{0}\}$.

Not onto since dim(col(A)) $= 2 \neq 3 =$ dim(codomain).

4. (14 points)

(a) Let A be any square matrix A. Show that the set S consisting of the vectors $\vec{v}$ that are fixed by the matrix A is a subspace, i.e. show that the set of vectors $\vec{v}$ such that $A\vec{v} = \vec{v}$ is a subspace.

Method 1:

$$A\vec{v} = \vec{v}$$
$$A\vec{v} - \vec{v} = \vec{0}$$
$$A\vec{v} - I\vec{v} = \vec{0}$$
$$(A-I)\vec{v} = \vec{0}$$

The set of vectors satisfying $A\vec{v} = \vec{v}$ is the same as the set of vectors satisfying $(A-I)\vec{v} = \vec{0}$, which is null$(A-I)$, hence it is a subspace (since null spaces are subspaces!)

Method 2: We can also show S is a subspace by showing it satisfies the definition.

① $\vec{0}$ is in S: $A\vec{0} = \vec{0}$, $\lambda\vec{0} = \vec{0}$ $\Rightarrow A\vec{0} = \lambda\vec{0}$, so $\vec{0}$ satisfies the equation. ✓

② If $\vec{u}$ and $\vec{v}$ are in S, is $\vec{u}+\vec{v}$ in S?
Since $\vec{u}$ and $\vec{v}$ are in S, $A\vec{u} = \lambda\vec{u}$ and $A\vec{v} = \lambda\vec{v}$.
Want to show: $A(\vec{u}+\vec{v}) = \lambda(\vec{u}+\vec{v})$. To do this:
$A(\vec{u}+\vec{v}) = A\vec{u} + A\vec{v} = \lambda\vec{u} + \lambda\vec{v} = \lambda(\vec{u}+\vec{v})$.
Thus, $\vec{u}+\vec{v}$ satisfies the equation. ✓

③ If $\vec{u}$ is in S and r is any scalar, is $r\vec{u}$ in S?
Since $\vec{u}$ is in S, $A\vec{u} = \lambda\vec{u}$. Want to show: $A(r\vec{u}) = \lambda(r\vec{u})$
$A(r\vec{u}) = rA\vec{u} = r(\lambda\vec{u}) = \lambda(r\vec{u})$.
Thus, $r\vec{u}$ satisfies the equation also. ✓

(b) Now, let

$$A = \begin{bmatrix} 3 & -2 & 3 \\ 4 & -3 & 6 \\ 2 & -2 & 4 \end{bmatrix}.$$

Give a basis for S as given in part (a). You may use the back of this page if you need more space.

Method 1: Find S, we saw before that $S = \text{null}(A-I)$ because:

$$S = \{\vec{v} \in \mathbb{R}^3 \mid A\vec{v} = \vec{v}\} = \{\vec{v} \in \mathbb{R}^3 \mid A\vec{v} - \vec{v} = \vec{0}\} = \{\vec{v} \in \mathbb{R}^3 \mid (A-I)\vec{v} = \vec{0}\}.$$

Directly compute: $(A-I)$

$$\begin{bmatrix} 3 & -2 & 3 \\ 4 & -3 & 6 \\ 2 & -2 & 4 \end{bmatrix} - \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 2 & -2 & 3 \\ 4 & -4 & 6 \\ 2 & -2 & 3 \end{bmatrix}$$

Now find solutions to $(A-I)\vec{v} = \vec{0}$.

$$\left[\begin{array}{ccc|c} 2 & -2 & 3 & 0 \\ 4 & -4 & 6 & 0 \\ 2 & -2 & 3 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} 2 & -2 & 3 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \rightarrow \begin{array}{l} v_3 = t \\ v_2 = s \\ v_1 = s - \frac{3}{2}t \end{array}$$

So $\vec{v} = s\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + t\begin{bmatrix} -\frac{3}{2} \\ 0 \\ 1 \end{bmatrix}$ and since lin. ind $\Rightarrow \mathcal{B}_S = \left\{\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -3/2 \\ 0 \\ 1 \end{bmatrix}\right\}$

Method 2: (Follow your nose approach.) Want to solve $A\vec{v} = \vec{v}$. We could try to put this into an augmented matrix:

$$\left[\begin{array}{ccc|c} 3 & -2 & 3 & v_1 \\ 4 & -3 & 6 & v_2 \\ 2 & -2 & 4 & v_3 \end{array}\right] \sim \ldots$$

but wait. (If you want to see this method, check out solutions to my Fall 2019 Midterm 2.)

what is this saying? Each line is an equation... in fact $A\vec{v}=\vec{v}$ is a system of equations. The issue is we have variables on both sides of the equation that we are solving for!

write out the system:

$$\begin{cases} 3v_1 - 2v_2 + 3v_3 = v_1 \\ 4v_1 - 3v_2 + 6v_3 = v_2 \\ 2v_1 - 2v_2 + 4v_3 = v_3 \end{cases} \xrightarrow{\text{combine variables}} \begin{cases} 2v_1 - 2v_2 + 3v_3 = 0 \\ 4v_1 - 4v_2 + 6v_3 = 0 \\ 2v_1 - 2v_2 + 3v_3 = 0 \end{cases}$$

[compare this to $A - I$!!!]

solve this homogeneous system!

$$\left[\begin{array}{ccc|c} 2 & -2 & 3 & 0 \\ 4 & -4 & 6 & 0 \\ 2 & -2 & 3 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} 2 & -2 & 3 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

So S is the set of all $\vec{v}$ of the form

$$v_3 = t, \quad v_2 = s, \quad v_1 = s - \tfrac{3}{2}t \implies \vec{v} = s\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + t\begin{bmatrix} -\frac{3}{2} \\ 0 \\ 1 \end{bmatrix}.$$

This is a solution to a homogeneous system, so these two vectors are linearly indepent, and $S = \operatorname{span}\left\{\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -\frac{3}{2} \\ 0 \\ 1 \end{bmatrix}\right\}$, so

$$\mathcal{B}_S = \left\{\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -\frac{3}{2} \\ 0 \\ 1 \end{bmatrix}\right\}.$$

Bonus: (5 points) A quadratic form is a function $Q(\vec{x})$ that satisfies the following two properties:

(a) $Q(r\vec{x}) = r^2 Q(\vec{x})$ for any vector $\vec{x}$ and r any real number (scalar).

(b) Fix a vector $\vec{y}$. Then $T(\vec{x}) = Q(\vec{x}+\vec{y}) - Q(\vec{x}) - Q(\vec{y})$ is a degree one polynomial in the entries of $\vec{x}$ (i.e. T is a linear equation).

Show that for a 2×2 matrix A, the function $Q : \mathbb{R}^2 \to \mathbb{R}$ given by

$$Q(\vec{x}) = \vec{x}^T A\vec{x}$$

is a quadratic form. (Note: $\vec{x}^T$ is a row vector.)

For (a), we can show this for any $n\times n$ matrix! (Same for (b) actually!)

$$Q(rx) = (r\vec{x})^T A (r\vec{x}) = r^2(\vec{x}^T A\vec{x}) = r^2 Q(\vec{x}).$$

Method 1: For (b), let $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$, $\vec{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ and $\vec{y} = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$.

Then $T(\vec{x}) = Q(\vec{x}+\vec{y}) - Q(\vec{x}) - Q(\vec{y})$

$$= (\vec{x}+\vec{y})^T A(\vec{x}+\vec{y}) - \vec{x}^T A\vec{x} - \vec{y}^T A\vec{y}$$

$$= \begin{bmatrix} x_1+y_1 \\ x_2+y_2 \end{bmatrix}^T \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}\begin{bmatrix} x_1+y_1 \\ x_2+y_2 \end{bmatrix} - [x_1\ x_2]\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} - [y_1\ y_2]\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}\begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$$

$$= \underbrace{[x_1+y_1\ \ x_2+y_2]\begin{bmatrix} a_{11}(x_1+y_1)+a_{12}(x_2+y_2) \\ a_{21}(x_1+y_1)+a_{22}(x_2+y_2) \end{bmatrix}}_{} - [x_1\ x_2]\begin{bmatrix} a_{11}x_1+a_{12}x_2 \\ a_{21}x_1+a_{22}x_2 \end{bmatrix} - [y_1\ y_2]\begin{bmatrix} a_{11}y_1+a_{12}y_2 \\ a_{21}y_1+a_{22}y_2 \end{bmatrix}$$

1x1 matrix, so I left off the brackets!

$$= a_{11}(x_1+y_1)^2 + a_{12}(x_2+y_2)(x_1+y_1) + a_{21}(x_1+y_1)(x_2+y_2) + a_{22}(x_2+y_2)^2$$

$$-(a_{11}x_1^2 + a_{12}x_1x_2 + a_{21}x_1x_2 + a_{22}x_2^2) - (a_{11}y_1^2 + (a_{12}+a_{21})y_1y_2 + a_{22}y_2^2)$$

Cancel terms!

$$= a_{12}x_2y_1 + a_{12}y_2x_1 + a_{21}x_1y_2 + a_{21}y_1x_2$$

$$= (a_{12}y_2 + a_{21}y_2)x_1 + (a_{12}y_1 + a_{21}y_1)x_2.$$

← linear!

(See next page for Method 2)

METHOD 2: $T(\vec{x}) = Q(\vec{x}+\vec{y}) - Q(\vec{x}) - Q(\vec{y})$

$$= (\vec{x}+\vec{y})^T A (\vec{x}+\vec{y}) - \vec{x}^T A \vec{x} - \vec{y}^T A \vec{y}$$

$$= (\vec{x}^T+\vec{y}^T)(A\vec{x}+A\vec{y}) - \vec{x}^T A \vec{x} - \vec{y}^T A \vec{y}$$

$$= \vec{x}^T A \vec{x} + \vec{y}^T A \vec{x} + \vec{x}^T A \vec{y} + \vec{y}^T A \vec{y} - \vec{x}^T A \vec{x} - \vec{y}^T A \vec{y}$$

$$= \vec{x}^T A \vec{y} + \vec{y}^T A \vec{x}.$$

Now, we can show T is a linear transformation, i.e. linear in the entries of $\vec{x}$!

① $T(r\vec{x}) = (r\vec{x})^T A \vec{y} + \vec{y}^T A (r\vec{x}) = r(\vec{x}^T) A \vec{y} + r\, \vec{y}^T A \vec{x} = r\left(\vec{x}^T A \vec{y} + \vec{y}^T A \vec{x}\right)$

$$= r\, T(\vec{x}).$$

② $T(\vec{x}+\vec{w}) = (\vec{x}+\vec{w})^T A \vec{y} + \vec{y}^T A (\vec{x}+\vec{w}) = (\vec{x}^T+\vec{w}^T) A \vec{y} + \vec{y}^T A \vec{x} + \vec{y}^T A \vec{w}$

$$= \vec{x}^T A \vec{y} + \vec{w}^T A \vec{y} + \vec{y}^T A \vec{x} + \vec{y}^T A \vec{w}$$

$$= \left[\vec{x}^T A \vec{y} + \vec{y}^T A \vec{x}\right] + \left[\vec{w}^T A \vec{y} + \vec{y}^T A \vec{w}\right]$$

$$= T(\vec{x}) + T(\vec{w}). \quad \boxtimes$$