MATH 308 O
Final Exam
June 8, 2020

Name ______________________________

Student ID #______________

HONOR STATEMENT

"I affirm that my work upholds the highest standards of honesty and academic integrity at the University of Washington, and that I have neither given nor received any unauthorized assistance on this exam."

SIGNATURE:______________________________

SOLUTIONS!

1	16	
2	8	
3	18	
4	12	
5	16	
Bonus	5	
Total	70	

- Your exam should consist of this cover sheet, followed by 4 problems and a bonus question. Check that you have a complete exam.
- Pace yourself. You have 110 minutes to complete the exam and there are 5 problems. Try not to spend more than 20 minutes on each problem. You will have 10 minutes at the end of the exam to upload your solutions to Gradescope.
- Show all your work and justify your answers.
- Your answers should be exact values rather than decimal approximations. (For example, $\frac{\pi}{4}$ is an exact answer and is preferable to its decimal approximation 0.7854.)
- This is an open book exam, however, you are not allowed to collaborate with anyone.
- There are multiple versions of the exam, you have signed an honor statement, and cheating is a hassle for everyone involved. DO NOT CHEAT.
- Turn your cell phone OFF and put it AWAY for the duration of the exam.

GOOD LUCK!

1. **Construct examples.** If you are asked to provide an example and there is no such example, write NOT POSSIBLE. No justification required.

   (a) (2 points) **Give an example** of a matrix A that represents the following transformation.

$$T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} xy \\ y \end{bmatrix}.$$

Not possible.

   (b) (2 points) **Give an example** of a set of linearly dependent vectors in $\mathbb{R}^3$ such that when you remove **any one** of the vectors, the remaining set is linearly independent and spans $\mathbb{R}^3$.

$$\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right\}$$

   (c) (2 points) **Give an example** of a 3×3 matrix A with eigenvalues 1 and -4, where $\text{rank}(A) = 2$.

$$A = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -4 \end{bmatrix}$$

Short Answer Questions.

(d) (4 points) Let $\vec{x} = \begin{bmatrix} 1 \\ 4 \end{bmatrix}$. Find a basis $\mathscr{B}$ such that $[\vec{x}]_{\mathscr{B}} = \begin{bmatrix} -1 \\ 2 \end{bmatrix}$. **Reminder:** A basis is a *set of vectors*, not a matrix.

Want: $\mathscr{B} = \{\vec{u}_1, \vec{u}_2\}$, where for $U = [\vec{u}_1\ \vec{u}_2]$, $\boxed{\vec{x} = U[\vec{x}]_{\mathscr{B}}.}$ *

Let $U = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. Then $\begin{bmatrix} 1 \\ 4 \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} -1 \\ 2 \end{bmatrix} = \begin{bmatrix} -a+2b \\ -c+2d \end{bmatrix}$

so we need $\begin{cases} -a+2b = 1 \\ -c+2d = 4. \end{cases}$ * Many choices! Pick one!

Let $b=0$, then $a=-1$ $\Rightarrow U = \begin{bmatrix} -1 & 0 \\ -2 & 1 \end{bmatrix}$

Let $d=1$, then $c=-2$ $\boxed{\text{and } \mathscr{B} = \left\{\begin{bmatrix} -1 \\ -2 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}\right\}}$

(e) (6 points) **Fill in the blanks.** Assume $p(\lambda) = (\lambda)(\lambda+2)^2(\lambda-2)(\lambda+4)^3$ is the characteristic polynomial of a matrix A. Then

i. A is a __7__ $\times$ __7__ matrix.

ii. The eigenvalues of A are __0, -2, 2, -4__

iii. Is A invertible? **Justify your answer.**

No, $\lambda = 0$ is an eigenvalue so by the unifying theorem, A is not invertible.

iv. Is A guaranteed to be diagonalizable? If so, justify your answer. If not, explain what you would need to know to guarantee A is diagonalizable.

No, A is not guaranteed to be diagonalizable. To guarantee that A is diagonalizable, we would need to know that E_{-2} (eigenspace corresponding to eigenvalue -2) is 2 dimensional, and E_{-4} (eigenspace corresponding to eigenvalue -4) is 3 dimensional. (Recall, E_0, E_2 are necessarily one dimensional.) Then diagonalizability is guaranteed by a Theorem!

2. (8 Points) Let A and B be $n \times n$ matrices, and determine if the following sets are subspaces of $\mathbb{R}^n$.

(a) $S = \{\vec{v} \in \mathbb{R}^n : A^2\vec{v} = AB\vec{v}\}$

$S = \{\vec{v} \in \mathbb{R}^n : A^2\vec{v} - AB\vec{v} = \vec{0}\}$

$S = \{\vec{v} \in \mathbb{R}^n : (A^2 - AB)\vec{v} = \vec{0}\}$

So, S is the null space of $(A^2 - AB)$, and by a theorem, since null spaces are subspaces, S is a subspace.

(b) $S = \{\vec{v} \in \mathbb{R}^n : A^2\vec{v} - I_n = \vec{0}\}$ ← $S = \phi$ (nothing is in S!)

$S = \{\vec{v} \in \mathbb{R}^n : \boxed{A^2\vec{v} = I_n}\}$

$\vec{0} \notin S$ since $A^2\vec{0} = \vec{0} \neq I_n$.

Thus, S is <u>not</u> a subspace.

(Alternatively)

$S = \{\vec{v} \in \mathbb{R}^n : A^2\vec{v} - \vec{e}_1 = \vec{0}\}$

$\vec{0} \notin S$ since $A \cdot \vec{0} \neq \vec{e}_1$.

3. (a) (2 points) Produce a 2×2 matrix that reflects $\mathbb{R}^2$ over the y-axis. Call this matrix S.

$\checkmark \; [T(\vec{e}_1) \; T(\vec{e}_2)]$

$$S = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}.$$

(b) (2 points) Produce a 2×2 matrix that rotates $\mathbb{R}^2$ by 90 degrees ($\frac{\pi}{2}$ radians) counter-clockwise. Call this matrix R.

$$\text{rot}\left(\tfrac{\pi}{2}\right) = \begin{bmatrix} \cos(\frac{\pi}{2}) & -\sin(\frac{\pi}{2}) \\ \sin(\frac{\pi}{2}) & \cos(\frac{\pi}{2}) \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

so $$R = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

(c) (2 points) Compute the matrix that represents a reflection of $\mathbb{R}^2$ over the y-axis then a rotation by 90 degrees counter-clockwise, in that order. Call this matrix C.

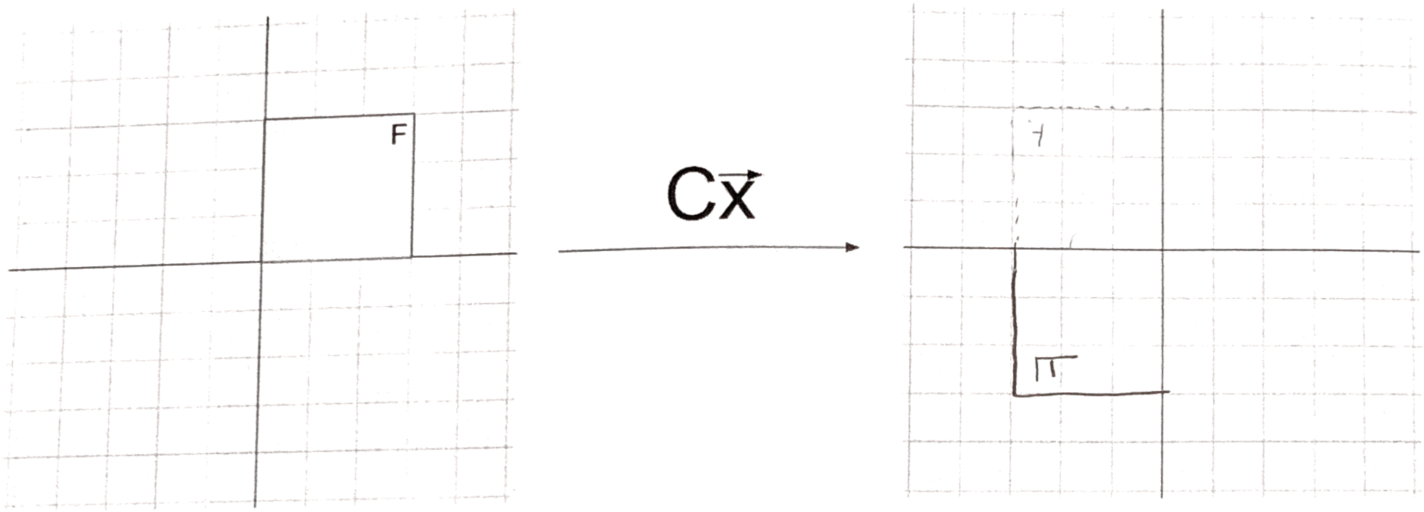
$$\underbrace{\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}}_{\substack{\text{rot ccw} \\ R}} \underbrace{\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}}_{\substack{\text{ref} \\ S}} = \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix} =: C$$

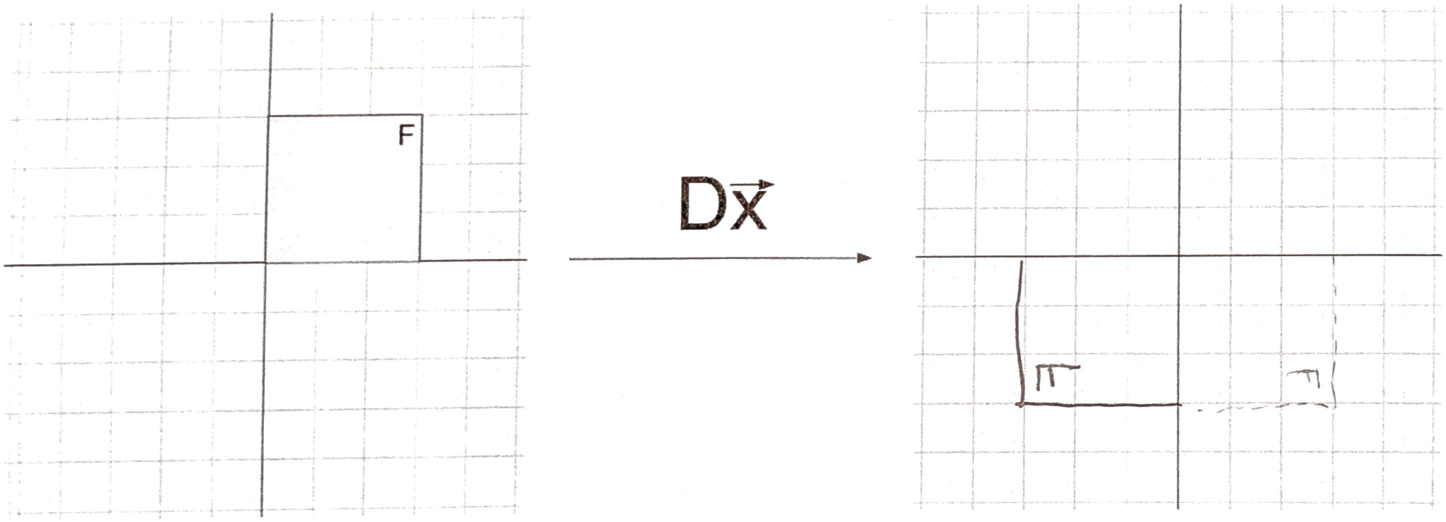
(d) (2 points) Compute the matrix that represents a rotation of $\mathbb{R}^2$ by 90 degrees **clockwise**, then a reflection over the y-axis, in that order. Call this matrix D.

Clockwise rotation by 90° : $R^{-1} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$

$$\underbrace{\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}}_{S} \underbrace{\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}}_{\substack{\text{rot cw} \\ R^{-1}}} = \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix} =: D$$

(e) (4 points) Complete the following drawings. Show where the unit square gets mapped and draw F with the correct orientation on the new square. Matrix C denotes the matrix from part (c), and matrix D denotes the matrix from part (d).





(f) (1 point) What relationship do C and D have? What does that mean about S and R? Express the relationship between S and R.

$C = D$, so $RS = SR^{-1}$

(g) (1 point) What happens if you apply the matrix S twice? Use geometric intuition first and write out what you think will happen, then compute S^2 to justify.

S is a reflection, so we should get the identity. (Reflecting twice takes you back to where you started!)

$$S^2 = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}^2 = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \checkmark$$

(h) (4 points) Use part (f) and part (g) to simplify the following expression as much as possible: $SRSRSRSRSRSRS$.

$= S\,SR^{-1}RS\,SR^{-1}RS\,SR^{-1}RS$

$= SSSSSSS$

$= S^2 S^2 S^2 S$, $\quad S^2 = I_2$

$\boxed{= S}$

2pts for responses that correctly end an incorrect part c, d, or f

4. Let A be a 3×3 matrix that satisfies the equation

$$A^3 + 2A^2 - I = 0$$

(a) (4 points) Show that the matrices A and $A + 2I$ are invertible.

Method 1: $A^3 + 2A^2 - I = 0$

(many solutions!)

$A^3 + 2A^2 = I$

$\underbrace{A}\underbrace{(A^2 + 2A)} = \underline{\underline{I}}$

$\Downarrow$

A is invertible and $A^{-1} = A^2 + 2A$ is the unique inverse!

$\underbrace{A^2}\underbrace{(A + 2I)} = \underline{\underline{I}}$

$\Downarrow$

$(A+2I)$ is invertible and A^2 is the unique inverse!

Method 2: $A^3 + 2A^2 = I$

$A^2(A+2I) = I$

so: $\det(A^2(A+2I)) = \det(I)$

$\det(A^2)\det(A+2I) = 1$

$\det(A)\det(A)\det(A+2I) = 1$

$\det(A) \neq 0$ and $\det(A+2I) \neq 0$, so by the unifying theorem, A and $A+2I$ are invertible.

(b) (4 points) If $\det(A) = \sqrt{3}$, what is the $\det(A + 2I)$?

Since $A^2(A+2I) = I$, as above in method 2,

$\det(A)\det(A)\det(A+2I) = \det(I) = 1$

$(\sqrt{3})(\sqrt{3})\det(A+2I) = 1$

$\det(A+2I) = \frac{1}{3}$

(many ways to get credit! this is just an exhaustive version!)

(c) (4 points) Explain why -2 is not an eigenvalue of A.

If -2 were an eigenvalue, then there would necessarily be a non-zero eigenvector $\vec{v}$ such that $A\vec{v} = -2\vec{v}$, But this means

$A\vec{v} + 2\vec{v} = \vec{0}$

$A\vec{v} + 2I\vec{v} = \vec{0}$

$(A+2I)\vec{v} = \vec{0}$

has non-trivial solutions, so $\text{null}(A+2I) \neq \{\vec{0}\}$. This means $A+2I$ is not invertible, but by applying the Unifying theorem we see this contradicts part (b) since $\det(A+2I) = \frac{1}{3} \neq 0$.

5. A linear transformation $T :\to \mathbb{R}^4 \to \mathbb{R}^3$ has the following properties:

- The vector $\begin{bmatrix}1\\0\\2\end{bmatrix}$ is in the range(T), but $\begin{bmatrix}0\\-1\\1\end{bmatrix}$ is not.
- The vectors $\vec{v}_1 = \begin{bmatrix}1\\0\\2\\0\end{bmatrix}$ and $\vec{v}_2 = \begin{bmatrix}-1\\1\\0\\0\end{bmatrix}$ both satisfy $T(\vec{v}_1) = T(\vec{v}_2) = \begin{bmatrix}2\\1\\2\end{bmatrix}$.

Answer the following questions about T. **Justify your answers.** And **do not** try to find the matrix representing T.

(a) (1 point) Is T one-to-one?

No, $\vec{v}_1$ and $\vec{v}_2$ map to the same vector.

(b) (1 point) Is T onto?

No, $\begin{bmatrix}0\\-1\\1\end{bmatrix}$ is not in range(T), so it cannot be onto.

(c) (2 points) Determine the dim(range(T)).

Dim(range(T)) $\neq 3$ since T is not onto. It is at least 2 since $\begin{bmatrix}1\\0\\2\end{bmatrix}$ and $\begin{bmatrix}2\\1\\2\end{bmatrix}$ are both in range(T), and form a linearly independent set:

$$\left[\begin{array}{cc|c}1&2&0\\0&1&0\\2&2&0\end{array}\right] \sim \left[\begin{array}{cc|c}1&2&0\\0&1&0\\0&-2&0\end{array}\right] \sim \left[\begin{array}{cc|c}1&2&0\\0&1&0\\0&0&0\end{array}\right] \checkmark$$

Thus, dim(range(T)) = 2.

(d) (2 points) Find a basis for range(T).

$\mathcal{B}_{\text{range}(T)}$ must contain 2 elements since $\dim(\text{range}(T)) = 2$.

Since $\left\{ \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix} \right\}$ is a linearly independent set (see part c), we can pick

$$\mathcal{B}_{\text{range}(T)} = \left\{ \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix} \right\}.$$

(e) (3 points) Find a nonzero vector $\vec{x}$ such that $T(\vec{x}) = \vec{0}$.

Notice: $T(\vec{v}_1) - T(\vec{v}_2) = \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix} - \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix} = \vec{0}$

$= T(\vec{v}_1 - \vec{v}_2)$, so $T(\vec{v}_1 - \vec{v}_2) = \vec{0}$.

Pick $\vec{x} = \vec{v}_1 - \vec{v}_2 = \begin{bmatrix} 1 \\ 0 \\ 2 \\ 0 \end{bmatrix} - \begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \\ 2 \\ 0 \end{bmatrix}$

(f) (3 points) Is your answer from part (e) a basis for ker(T)?

No. The set $\left\{ \begin{bmatrix} 2 \\ -1 \\ 2 \\ 0 \end{bmatrix} \right\}$ is linearly independent (being non-zero), but by the rank-nullity Theorem, since $\dim(\text{range}(T)) = 2$, we have $\dim(\text{range}(T)) + \dim(\ker(T)) = 4$, so $\dim(\ker(T)) = 2$ and we see that we need at least two linearly independent vectors in a basis for $\ker(T)$.

(g) (4 points) Find another vector $\vec{w}$ that is not $\vec{v}_1$ or $\vec{v}_2$ such that $T(\vec{w}) = \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix}$.

(Many solutions! Here's one.)

Take $\vec{v}_1$ and add a vector from the null space.

(Recall, $\vec{x}_g = \vec{x}_p + \underbrace{\vec{x}_h}_{\text{null space!}}$) Since $\begin{bmatrix} 2 \\ -1 \\ 2 \\ 0 \end{bmatrix} \in \ker(T)$,

pick $\vec{w} = \begin{bmatrix} 1 \\ 0 \\ 2 \\ 0 \end{bmatrix} + \begin{bmatrix} 2 \\ -1 \\ 2 \\ 0 \end{bmatrix} = \underline{\underline{\begin{bmatrix} 3 \\ -1 \\ 4 \\ 0 \end{bmatrix}}}$.

(This works! $T\left(\vec{v}_1 + \begin{bmatrix} 2 \\ -1 \\ 2 \\ 0 \end{bmatrix}\right) = T(\vec{v}_1) + T\left(\begin{bmatrix} 2 \\ -1 \\ 2 \\ 0 \end{bmatrix}\right)$
$= T(\vec{v}_1) + \vec{0}$
$= \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix}$.)

BONUS: The transpose operation is a linear transformation on $n \times n$ matrices. By "linear transformation on $n \times n$ matrices" we mean that the transpose operation satisfies the two usual conditions in the definition, but instead of applying the transformation to a vector, we can apply it to a matrix.

(a) (1 point) Show that the transpose operation is a linear transformation on $n \times n$ matrices.

① $(B+A)^T = B^T + A^T$.

② $(rA)^T = r(A^T)$.

properties of the transpose.

(b) (4 points) Since the transpose is a linear transformation, we can find a matrix that represents it. However, that requires understanding that a collection of $m \times m$ matrices can be thought of as some $\mathbb{R}^n$. Find a matrix that represents this linear transformation on 2×2 matrices. **Hint:** Can you write the matrix as a vector somehow? You must choose a basis. (No credit will be awarded if it is not clear what basis you have chosen.)

Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. Then $A^T = \begin{bmatrix} a & c \\ b & d \end{bmatrix}$.

To think of A as a vector ... pick:

$$\mathcal{B}_{2\times 2} = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\},$$

basis for 2×2 matrices.

Then $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} = a\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + b\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} + c\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} + d\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$,

and $[A]_{\mathcal{B}_{2\times 2}} = \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix}$ represents the matrix. Then we want a linear transformation sending $[A]_{\mathcal{B}_{2\times 2}} = \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} \longrightarrow \begin{bmatrix} a \\ c \\ b \\ d \end{bmatrix} = [A^T]_{\mathcal{B}_{2\times 2}}$.

Pick:

$$M = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix},$$ then $M[A]_{\mathcal{B}_{2\times 2}} = [A^T]_{\mathcal{B}_{2\times 2}}$.