

## 12.4 Review

In 12.4, we learn about cross products.

- For two vectors  $\mathbf{a} = \langle a_1, a_2, a_3 \rangle$  and  $\mathbf{b} = \langle b_1, b_2, b_3 \rangle$ , we define

$$\mathbf{a} \times \mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} = \begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix} \mathbf{i} - \begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix} \mathbf{j} + \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} \mathbf{k}$$

which is equal to

$$\mathbf{a} \times \mathbf{b} = (a_2b_3 - a_3b_2)\mathbf{i} - (a_1b_3 - a_3b_1)\mathbf{j} + (a_1b_2 - a_2b_1)\mathbf{k}.$$

- The cross product is defined in this unusual way in order to accomplish one main goal (you can sort of see why it is defined this way or how you could reverse engineer where this came from when you see the goal). **The main fact about cross products for this class is:**

$$\mathbf{a} \times \mathbf{b} \text{ is orthogonal to both } \mathbf{a} \text{ and } \mathbf{b}.$$

You can see this is true simply by using what we learned about dot product from last section. Meaning the proof is just checking the two dot products below (this is how someone came up with the cross product, they successfully found a pattern that would always make this happen):

$$\mathbf{a} \cdot (\mathbf{a} \times \mathbf{b}) = a_1(a_2b_3 - a_3b_2) - a_2(a_1b_3 - a_3b_1) + a_3(a_1b_2 - a_2b_1) = 0$$

$$\mathbf{b} \cdot (\mathbf{a} \times \mathbf{b}) = b_1(a_2b_3 - a_3b_2) - b_2(a_1b_3 - a_3b_1) + b_3(a_1b_2 - a_2b_1) = 0.$$

Using the main fact, we can always check that we did the cross product correctly. After computing a cross product, ALWAYS go back and check that the two dot products above come out to be zero. If they don't both come out to be zero, then you made a mistake in your cross product. Don't ever skip this step, always check your work!

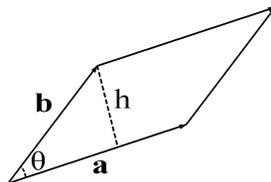
- Essential Properties. You should recognize that these properties essentially tell us that the cross product satisfies most of the standard rules that we are used to using for regular products (EXCEPT it is not commutative):

(a)  $\mathbf{a} \times \mathbf{b} = -\mathbf{b} \times \mathbf{a}$  (so if you switch the order you flip all the signs).

(b)  $\mathbf{a} \times (\mathbf{b} + \mathbf{c}) = \mathbf{a} \times \mathbf{b} + \mathbf{a} \times \mathbf{c}$

(c)  $c(\mathbf{a} \times \mathbf{b}) = (c\mathbf{a}) \times \mathbf{b} = \mathbf{a} \times (c\mathbf{b})$

- Interpreting the length of  $\mathbf{a} \times \mathbf{b}$ : Just like we did when interpreting the dot product, draw  $\mathbf{a}$  and  $\mathbf{b}$  tail to tail and label the angle between them  $\theta$  ( $0 \leq \theta \leq \pi$ ). But now draw the parallelogram that is formed by taking  $\mathbf{a}$  and  $\mathbf{b}$  as the sides. Also label the height of the parallelogram as  $h$ . (see below)



Using basic trigonometry, we see that  $h = |\mathbf{b}| \sin(\theta)$ . So the area of this parallelogram is given by  $|\mathbf{a}|h = |\mathbf{a}||\mathbf{b}| \sin(\theta)$ .

A fact that is not obvious, but can be seen by expanding out from the definition of cross product (see the textbook for this), is that  $(\mathbf{a} \times \mathbf{b}) \cdot (\mathbf{a} \times \mathbf{b}) = (a_1^2 + a_2^2 + a_3^2)(b_1^2 + b_2^2 + b_3^2) - (a_1b_1 + a_2b_2 + a_3b_3)^2$ , which can be written as  $|\mathbf{a} \times \mathbf{b}|^2 = |\mathbf{a}|^2|\mathbf{b}|^2 - (\mathbf{a} \cdot \mathbf{b})^2$ . We then use our main fact about dot products to get  $|\mathbf{a} \times \mathbf{b}|^2 = |\mathbf{a}|^2|\mathbf{b}|^2 - |\mathbf{a}|^2|\mathbf{b}|^2 \cos^2(\theta)$ . Factoring and using the identity  $1 - \cos^2(\theta) = \sin^2(\theta)$  gives  $|\mathbf{a} \times \mathbf{b}|^2 = |\mathbf{a}|^2|\mathbf{b}|^2 \sin^2(\theta)$ . Thus (since  $\sin(\theta)$  is positive for this range of  $\theta$  values), we have

$$|\mathbf{a} \times \mathbf{b}| = |\mathbf{a}||\mathbf{b}| \sin(\theta) = \text{'area of parallelogram'}$$

(Note that if the angle between the vectors is 90 degrees, then the parallelogram is just a rectangle and since  $\sin(\pi/2) = 1$  we get the correct 'length times width' formula for the area, namely  $|\mathbf{a} \times \mathbf{b}| = |\mathbf{a}||\mathbf{b}|$ )

5. Remember if  $\mathbf{a}$  and  $\mathbf{b}$  are parallel, then the angle between them is  $\theta = 0$  or  $\theta = \pi$ . Thus, an immediate consequence of the formula above is that:

$$\text{if } \mathbf{a} \text{ and } \mathbf{b} \text{ are parallel, then } |\mathbf{a} \times \mathbf{b}| = 0 \text{ which means } \mathbf{a} \times \mathbf{b} = \langle 0, 0, 0 \rangle.$$

But remember the best way to test if two vectors are parallel is to see if they are scalar multiples of each other.

6. Note that the area of the triangle formed by the two vectors  $\mathbf{a}$  and  $\mathbf{b}$  would be exactly half of the area of the parallelogram. Thus,

$$\text{'Area of the triangle formed by } \mathbf{a} \text{ and } \mathbf{b}\text{' } = \frac{1}{2}|\mathbf{a} \times \mathbf{b}|.$$

This is a nice, general, and efficient formula for finding the area of any triangle in  $\mathbb{R}^2$  or  $\mathbb{R}^3$  (for  $\mathbb{R}^2$  just make the third component zero).

7. Note there are infinitely many vectors which are orthogonal to two given nonzero vectors  $\mathbf{a}$  and  $\mathbf{b}$ . So the vector  $\mathbf{v} = \mathbf{a} \times \mathbf{b}$  is not the only vector orthogonal to both  $\mathbf{a}$  and  $\mathbf{b}$ . But we can say that all vectors that are orthogonal to both  $\mathbf{a}$  and  $\mathbf{b}$  are scale multiples of  $\mathbf{v}$ . Meaning any vector that is orthogonal to both  $\mathbf{a}$  and  $\mathbf{b}$  will be either in the same direction as  $\mathbf{a} \times \mathbf{b}$  or in the opposite direction.

For the applications in this particular class (mostly finding planes), we will encounter situations where we just need any vector that is orthogonal to both  $\mathbf{a}$  and  $\mathbf{b}$ , so we won't often care about the vectors length or direction. But in other classes where you encounter vectors, it may be important to know which direction  $\mathbf{a} \times \mathbf{b}$  points.

This direction is given to us by the so called 'right-hand rule' which says that if you curl the fingers of your right hand from  $\mathbf{a}$  to  $\mathbf{b}$ , then your thumb will point in the direction of  $\mathbf{a} \times \mathbf{b}$ . Here are a couple examples

- If  $\mathbf{a} = \langle 1, 0, 0 \rangle$  (a vector that points parallel to the  $x$ -axis) and  $\mathbf{b} = \langle 0, 1, 0 \rangle$  (a vector that points parallel to the  $y$ -axis), then you can verify through computation that  $\mathbf{a} \times \mathbf{b} = \langle 0, 0, 1 \rangle$  which points **upward**. The right-hand rule would have expected this upward direction because if you curl the fingers of your right hand from  $(1, 0, 0)$  on the  $x$ -axis toward  $(0, 1, 0)$  on the  $y$ -axis, then your thumb is pointing upward. (and if you did  $\mathbf{b} \times \mathbf{a}$  then your thumb would point downward and you get the vector  $\langle 0, 0, -1 \rangle$ )
- If  $\mathbf{a} = \langle 1, 1, 1 \rangle$  (a vector that points straight out in the first octant) and  $\mathbf{b} = \langle 0, 0, 3 \rangle$  (a vector that points upward parallel to the  $z$ -axis), then you can verify through computation

that  $\mathbf{a} \times \mathbf{b} = \langle 3, -3, 0 \rangle$ . This vector points in the positive  $x$ -direction, negative  $y$ -direction and parallel to the  $xy$ -plane.

The right-hand rule would have expect these directions since if you curl the fingers of your right hand from  $(1, 1, 1)$  toward  $(0, 0, 3)$ , then in the positive  $x$ -direction, negative  $y$ -direction, and is parallel to the  $xy$ -plane. (again, if you did  $\mathbf{b} \times \mathbf{a}$  then your thumb would point in the opposite direction).

Again, I wouldn't lose any sleep over the right-hand rule in this class, it's just a nice way to visually predict the direction of  $\mathbf{a} \times \mathbf{b}$  and you'll see it in future classes (mostly in physics).

8. ASIDE (for your own interest): The textbook mentions the so called scalar triple product (we skip this concept in this course, but you will see it if you take more vector calculus courses). If you take three vectors  $\mathbf{a}$ ,  $\mathbf{b}$  and  $\mathbf{c}$  in  $\mathbb{R}^3$  that aren't zero and with none of them parallel, then when they are all drawn tail to tail they form what is called a parallelepiped (a three dimensional parallelogram). We can find the volume of this three dimensional parallelepiped by using '(height) times (area of base)'. The base is a parallelogram with area  $|\mathbf{b} \times \mathbf{c}|$ . And since  $\mathbf{b} \times \mathbf{c}$  points perpendicular to the base, the height will be given by  $|\mathbf{a}| \cos(\theta)$  where  $\theta$  is the angle between  $\mathbf{a}$  and  $\mathbf{b} \times \mathbf{c}$  (we have to be a little careful here since this vector could point upward or downward which will make cosine of the angle positive or negative, but we will just take the absolute value in the next step to deal with this issue). Using the main fact about dot products, we get what is called the scalar triple product  $|\mathbf{a}| |\mathbf{b} \times \mathbf{c}| \cos(\theta) = \mathbf{a} \cdot (\mathbf{b} \times \mathbf{c})$  and the volume is the absolute value of this number, namely, 'volume of the parallelepiped' =  $|\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c})|$ .
9. ASIDE (for your own interest): In physics, the concept of torque concerns the tendency of an object to rotate typically from a force being applied by a wrench at some distance away. Assume a wrench is tightening a bolt. Find the vector  $\mathbf{r}$  that has its tail at the bolt and its head at the end of the wrench (this is the vector that represents how far away from the bolt the force is going to be applied). Then let  $\mathbf{F}$  be the force vector that represents the magnitude and direction of the force being applied at the end of the wrench. In physics, we define  $\tau = |\mathbf{r} \times \mathbf{F}|$  to be the magnitude of the torque and we say  $\boldsymbol{\tau} = \mathbf{r} \times \mathbf{F}$  is the torque vector. Note that if you draw  $\mathbf{r}$  and  $\mathbf{F}$  tail to tail and use the right hand rule, then you get the typical 'righty-tighty, lefty-loosy' rule (meaning if you are turning the bolt clockwise, then  $\boldsymbol{\tau}$  points down in the direction of the bolt being tightened and if you are turning the bolt counterclockwise, then  $\boldsymbol{\tau}$  points up in the direction of the bolt being loosened). This might help you better visualize the right-hand rule as the standard rule from tightening screws.