MATH 308 F
Final Exam
July 22, 2020

Name ____________________________________

Student ID #__________________

HONOR STATEMENT

"I affirm that my work upholds the highest standards of honesty and academic integrity at the University of Washington, and that I have neither given nor received any unauthorized assistance on this exam."

SIGNATURE:______________________________

SOLUTIONS!

1	9	
2	14	
3	12	
4	10	
5	6	
6	13	
7	16	
Bonus	8	
Total	80	

- Your exam should consist of this cover sheet, followed by 7 problems and a bonus. Check that you have a complete exam.
- Pace yourself. You have 120 minutes to complete the exam and there are 7 problems. Try not to spend more than 15 minutes on each problem. You will have 10 minutes at the end of the exam to upload your solutions to Gradescope.
- Show all your work and justify your answers.
- Your answers should be exact values rather than decimal approximations. (For example, $\frac{\pi}{4}$ is an exact answer and is preferable to its decimal approximation 0.7854.)
- This is an open book exam, however, you are not allowed to collaborate with anyone.
- There are multiple versions of the exam, you have signed an honor statement, and cheating is a hassle for everyone involved. DO NOT CHEAT.
- Turn your cell phone OFF and put it AWAY for the duration of the exam.

GOOD LUCK!

1. (9 Points) Clearly indicate whether the statement is true or false. **Justify your answer.** You may cite properties from theorems. Remember that for a statement to be true, it must be true under all possible scenarios (there can be no counterexample).

   (a) **TRUE / FALSE** $\mathbb{R}^4$ is a subspace of $\mathbb{R}^5$.

   False $\mathbb{R}^4$ is not a subset of $\mathbb{R}^5$.

   or

   $\begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \notin \mathbb{R}^4$ (does not satisfy condition 1)

   (b) **TRUE / FALSE** Let λ be an eigenvalue of an $n \times n$ matrix A. Then λ^4 is an eigenvalue of A^4.

   True If $A\vec{v} = \lambda\vec{v}$, $\vec{v}$ is an eigenvector corresponding to λ, then

   $A^4\vec{v} = A^3\lambda\vec{v} = \lambda A^3\vec{v} = \lambda^2 A^2\vec{v} = \lambda^3 A\vec{v} = \lambda^4\vec{v}$, and $\vec{v} \neq 0$.

   Since $A^4\vec{v} = \lambda^4\vec{v}$ and $\vec{v}$ is non-zero, $\vec{v}$ is an eigenvector with eigenvalue λ^4.

   (c) **TRUE / FALSE** For any $n \times n$ matrix A, if $\det(A) > 0$, then each $n-1 \times n-1$ minor is also positive.

   False Counter example: $A = \begin{bmatrix} 2 & -1 \\ 1 & 2 \end{bmatrix}$, $\det(A) = 5$

   but $M_{21} = [-1]$ $\left(\begin{bmatrix} \cancel{2} & -1 \\ \cancel{1} & \cancel{2} \end{bmatrix} \right)$

   and the minor, $\det(M_{21}) = -1$, which is not positive.

2. (14 points) Construct an example. If there is no such example, write NOT POSSIBLE **and justify your answer**. If you provide an example, you **do not** need to justify that your example satisfies the desired conditions, but do show all of your work.

(a) **Construct an example** of a 3×4 matrix A and a 4×3 matrix B such that the linear transformation represented by the matrix AB is onto.

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}_{3\times 4} \qquad \left(\text{Then } AB = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.\right)$$

$$B = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}_{4\times 3}$$

(b) **Construct an example** of a 3×4 matrix A and a 4×3 matrix B such that the linear transformation represented by the matrix BA is onto.

<u>NOT POSSIBLE</u> $A_{3\times 4}$, let $T_A : \mathbb{R}^4 \to \mathbb{R}^3$ be $T_A(\vec{x}) = A\vec{x}$

$B_{4\times 3}$, let $T_B : \mathbb{R}^3 \to \mathbb{R}^4$ be $T_B(\vec{x}) = B\vec{x}$.

Since $3 < 4$, T_B is not onto. This means, there exists a $\vec{y} \in \mathbb{R}^4$ such that $T_B(\vec{x}) \neq \vec{y}$ for any $\vec{x} \in \mathbb{R}^3$. Even if T_A is onto, $T_B \circ T_A(\vec{x}) = T_B(T_A(\vec{x})) \neq \vec{y}$.

(c) **Construct an example** of a 2×2 matrix with no real eigenvalues.

(e.g. rotation)

$$\text{rot}(\pi/2) = \begin{bmatrix} \cos\frac{\pi}{2} & -\sin\frac{\pi}{2} \\ \sin\pi/2 & \cos\pi/2 \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

$$\left(\text{Check: } \det(A - \lambda I) = \begin{vmatrix} -\lambda & -1 \\ 1 & -\lambda \end{vmatrix} = \lambda^2 + 1, \text{ no real eigenvalues!}\right)$$

(d) **Construct an example** of a linear transformation $T : \mathbb{R}^4 \to \mathbb{R}^4$ such that the kernel of T is the same set of vectors in $\mathbb{R}^4$ as the range of T.

(Construct a matrix A such that null(A) = Col(A).)

Pick null(A) to be span $\left\{ \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\}$;

$$\begin{bmatrix} & & 0 & 0 \\ & & 0 & 0 \\ & & 0 & 0 \\ & & 0 & 0 \end{bmatrix}$$

↑ ↑ f.v. f.v.

((Collapses x_3, x_4 coordinates!))

then pick $\text{Col}(A) = \left\{ \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\}$

$$A = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}.$$

Let $\underline{\underline{T(\vec{x}) = A\vec{x}}}$.

$\left(\text{Check: null}(A):\ \vec{x} = \begin{bmatrix} 0 \\ 0 \\ s \\ t \end{bmatrix} \checkmark \right)$

3. (12 Points) Let A be an $n \times n$ matrix with characteristic polynomial

$$p_A(\lambda) = (\lambda+1)^7(\lambda-2)^3(\lambda+4)(\lambda+5)$$

Answer the following questions with justification.

(a) What is n? (A is $n \times n$.)

$n=12$ (degree of the characteristic polynomial.)

(b) Compute the determinant of A.

Recall: $p_A(\lambda) = \det(A-\lambda I)$, so $p_A(0) = \det(A)$,

$p_A(0) = 1^7 \cdot (-2)^3 \cdot 4 \cdot 5 = -160$

(c) What is the largest possible value for rank$(A - 2I_n)$?

$\text{rank}(A-2I_n) = \dim(\text{col}(A-2I_n))$. Notice, $\dim(\text{null}(A-2I_n)) = \dim(E_2)$.
Since $\dim(E_2) \le$ multiplicity of 2, we know nullity$(A-2I_n) = \dim(E_2) \le 3$.
$\dim(E_2) \neq 0$ since eigenspaces must contain non-zero vectors. Thus, the smallest nullity of $(A-2I_n)$ would be 1. By rank-nullity, since rank$(A-2I_n)$ + nullity$(A-2I_n) = 12$, the largest rank would be $\underline{11}$.

(d) What is the smallest possible value for nullity$(A + 4I_n)$?

Notice that $\lambda=-4$ is an eigenvalue of multiplicity 1, and since nullity$(A+4I_n)$ $= \dim(\text{null}(A+4I_n)) = \dim(E_{-4}) \le$ multiplicity of $-4 = 1$, we see that nullity$(A+4I_n) \le 1$. However, nullity$(A+4I_n)$ cannot be 0 since the eigenspace must contain non-zero vectors. Thus, the smallest possible value for nullity$(A+4I_n)$ is $\underline{1}$.

(e) What is the largest possible value for nullity$(A - I_n)$?

The largest possible value for nullity of $A-I_n$ is $\underline{0}$, because otherwise, nullity$(A-I_n) = \dim(\text{null}(A-I_n)) > 0$, which means there exists non-zero vectors $\vec{v} \in \text{null}(A-I_n)$, which means $\lambda = 1$ is an eigenvalue, but we can tell it is not by looking at $p_A(\lambda)$.

(f) Let B be an $n \times n$ matrix such that the value of its characteristic polynomial $p_B(\lambda)$ evaluated at 0 is -160. In other words, $p_B(0) = -160$. Is it possible that $B = A^T$? Why or why not?

From part b, recall that $p_B(0) = -160 = \det(B)$. Since $\det(A) = -160$, and $\det(A) = \det(A^T)$, it is possible.

4. Let $A = [\vec{a}_1 \ \vec{a}_2 \ \vec{a}_3 \ \vec{a}_4]$ be a matrix equivalent to the following matrix:

$$\begin{array}{c} \begin{bmatrix} 1 & -1 & 1 & -3 \\ 0 & 0 & 2 & -6 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \\ \begin{array}{cccc} \uparrow\,\text{p.c.} & \quad & \uparrow\,\text{p.c.} & \quad \end{array} \end{array}$$

(a) (2 points) Generate a basis for the column space of A.

Method 2: $\mathcal{B}_{col(A)} = \{\vec{a}_1, \vec{a}_3\}$

pick columns in A corresponding to pivots in the echelon form matrix.

(b) (2 points) Generate a basis for the row space of A.

Method 1: $\mathcal{B}_{row(A)} = \left\{ \begin{bmatrix} 1 \\ -1 \\ 1 \\ -3 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 2 \\ -6 \end{bmatrix} \right\}$

Pick non-zero rows in echelon form matrix.

(c) (6 points) Generate a basis for the column space of A that includes the vector $\vec{a_1} + \vec{a_3}$. Show all of your steps and justify your answer. If this is not possible, explain why.

First, notice $\vec{a}_1 + \vec{a}_3 \in col(A)$ since it is a linear combination of the column vectors. Next, notice $\mathcal{B}_{col(A)} = \{\vec{a}_1, \vec{a}_3\}$ from part a tells us $\dim(col(A)) = 2$.

So, we need two linearly independent vectors, and by a Theorem, we know this will be a basis.

Many solutions!

Claim: $\{\vec{a}_1, \vec{a}_1 + \vec{a}_3\}$ is a linearly independent set.

$x_1 \vec{a}_1 + x_2(\vec{a}_1 + \vec{a}_3) = \vec{0}$

Thus, $\{\vec{a}_1, \vec{a}_1 + \vec{a}_3\}$ is a basis for $col(A)$.

$(x_1 + x_2)\vec{a}_1 + x_2 \vec{a}_3 = \vec{0}$

↳ [Since 1st and 3rd columns are linearly independent in $\begin{bmatrix} 1 & -1 & 1 & -3 \\ 0 & 0 & 2 & -6 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$, we know: $x_2 = 0$ $x_1 + x_2 = 0 \Rightarrow x_1 = 0$.

5. (6 points) For what values of c is the following set of vectors linearly dependent? For what values of c does the set span $\mathbb{R}^3$? Justify your answer.

$$\vec{v_1} = \begin{bmatrix} 1 \\ 8 \\ c \end{bmatrix}, \vec{v_2} = \begin{bmatrix} 1 \\ c \\ 0 \end{bmatrix}, \vec{v_3} = \begin{bmatrix} 0 \\ c \\ 4 \end{bmatrix}.$$

By the Unifying Theorem, since $\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ is a set of 3 vectors in $\mathbb{R}^3$, we know that the set is linearly dependent if and only if the determinant of $A = [\vec{v}_1 \ \vec{v}_2 \ \vec{v}_3]$ is 0. Similarly, the set will span $\mathbb{R}^3$ if and only if the determinant of $A = [\vec{v}_1 \ \vec{v}_2 \ \vec{v}_3]$ is not 0.

$$A = \begin{bmatrix} 1 & 1 & 0 \\ 8 & c & c \\ c & 0 & 4 \end{bmatrix}, \quad \det(A) = 4c + c^2 + 0 - 0 - 32 - 0$$

$$= c^2 + 4c - 32$$

$$= (c+8)(c-4)$$

So $\det(A) = 0$ iff $c = 4, -8$ and $\det(A) \neq 0$ iff $c \neq 4, -8$.

Thus, by the unifying theorem, the set is linearly dependent if and only if $c = 4, -8$ and the set will span $\mathbb{R}^3$ iff $c \neq 4, -8$.

6. (13 points) Let $p(x) = ax^2+bx+c$ denote an arbitrary polynomial of degree 2, with constants a, b, and c. To each such polynomial, associate the vector

$$ax^2 + bx + c \longleftrightarrow \begin{bmatrix} a \\ b \\ c \end{bmatrix} \in \mathbb{R}^3$$

Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be the transformation $T(p(x)) = p'(x)$.

(a) Find a matrix A that represents the linear transformation T, i.e. so that $T(\vec{x}) = A\vec{x}$. (If this is possible, this means that T is a linear transformation.)

Rem: $p(x) = ax^2+bx+c \Rightarrow p'(x) = 2ax+b$

So $T\left(\begin{bmatrix} a \\ b \\ c \end{bmatrix}\right) = \begin{bmatrix} 0 \\ 2a \\ b \end{bmatrix}$

$$\boxed{\text{Pick } A = \begin{bmatrix} 0 & 0 & 0 \\ 2 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}.}$$

Then $A\begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 2 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}\begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 0 \\ 2a \\ b \end{bmatrix}$.

(b) Is T one-to-one? If so, justify your answer. If not, find two polynomials that get sent to the same polynomial.

T is not one-to-one. Let $p(x) = x^2+1$, $q(x) = x^2+2$.

(Then $p'(x) = 2x = q'(x)$.)

$T(p(x)) = T\left(\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 0 & 0 & 0 \\ 2 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 2 \\ 0 \end{bmatrix}$

$T(q(x)) = T\left(\begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}\right) = \begin{bmatrix} 0 & 0 & 0 \\ 2 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} = \begin{bmatrix} 0 \\ 2 \\ 0 \end{bmatrix}$

$\Rightarrow T(p(x)) = T(q(x))$

but $p(x) \neq q(x)$.

(c) Is T onto? If so, justify your answer. If not, find a polynomial that is not in the range of T.

No, T is not onto. $p(x) = x^2$ is not in the range of T:

$T(\vec{x}) = A\vec{x}$, $A\begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 0 \\ 2a \\ b \end{bmatrix}$

cannot pick $\begin{bmatrix} a \\ b \\ c \end{bmatrix}$ so that $A\begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$.

(d) What is A^2? A^3? A^4? How does that relate to the idea of a derivative? Explain your answer.

$A^2 = \begin{bmatrix} 0 & 0 & 0 \\ 2 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}\begin{bmatrix} 0 & 0 & 0 \\ 2 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 2 & 0 & 0 \end{bmatrix}$ $\rightsquigarrow$ $\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 2 & 0 & 0 \end{bmatrix}\begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 2a \end{bmatrix}$

$p(x) = ax^2 + bx + c$

$p'(x) = 2ax + b$

$\boxed{p''(x) = 2a}$

$A^3 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 2 & 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & 0 & 0 \\ 2 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ $\leftarrow$ $\boxed{p'''(x) = 0}$

$A^4 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ $\leftarrow$ $\boxed{p^{4}(x) = 0}$

(*) Each application of A is like taking a derivative, So A^2 gives the 2^{nd} derivative, A^3 the third, and A^4 the fourth.

(e) Describe the behavior of A geometrically. What does it do to the vectors in $\mathbb{R}^3$?

Many possible answers!

e.g. A projects and then stretches one coordinate.

e.g. A sends $\vec{e}_1$ to $2\vec{e}_2$, $\vec{e}_2$ to $\vec{e}_3$, and $\vec{e}_3$ to $\vec{0}$.

7. (16 points)

(a) Find the all eigenvalues of the matrix $A = \begin{bmatrix} 3 & -1 \\ -1 & 3 \end{bmatrix}$.

$\det \begin{vmatrix} 3-\lambda & -1 \\ -1 & 3-\lambda \end{vmatrix} = (3-\lambda)^2 - 1 = 9 - 6\lambda + \lambda^2 - 1 = \lambda^2 - 6\lambda + 8 = (\lambda-4)(\lambda-2) = 0$

Eigenvalues: $\lambda = 4$, $\lambda = 2$

(b) Find an eigenvector corresponding to each eigenvalue of the matrix.

$\lambda = 4$: $(A - 4I_2)\vec{v} = \vec{0}$

$\left[\begin{array}{cc|c} 3-4 & -1 & 0 \\ -1 & 3-4 & 0 \end{array}\right] = \left[\begin{array}{cc|c} -1 & -1 & 0 \\ -1 & -1 & 0 \end{array}\right] \sim \left[\begin{array}{cc|c} -1 & -1 & 0 \\ 0 & 0 & 0 \end{array}\right]$

$x_2 = t$
$x_1 = -t$ $\Rightarrow \vec{v} = t\begin{bmatrix} -1 \\ 1 \end{bmatrix} \left(= \text{span}\left\{\begin{bmatrix} -1 \\ 1 \end{bmatrix}\right\}\right)$

$\boxed{\text{Pick } \vec{v}_1 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}.}$

$\lambda = 2$: $(A - 2I_2)\vec{v} = \vec{0}$

$\left[\begin{array}{cc|c} 3-2 & -1 & 0 \\ -1 & 3-2 & 0 \end{array}\right] = \left[\begin{array}{cc|c} 1 & -1 & 0 \\ -1 & 1 & 0 \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & -1 & 0 \\ 0 & 0 & 0 \end{array}\right]$

$x_2 = t$
$x_1 = t$ $\Rightarrow \vec{v} = t\begin{bmatrix} 1 \\ 1 \end{bmatrix} \left(= \text{span}\left\{\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right\}\right)$

$\boxed{\text{Pick } \vec{v}_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}.}$

(c) Express each eigenspace as of the span of a set of vectors.

$\lambda = 4$: $E_4 = \text{span}\left\{\begin{bmatrix} -1 \\ 1 \end{bmatrix}\right\}$

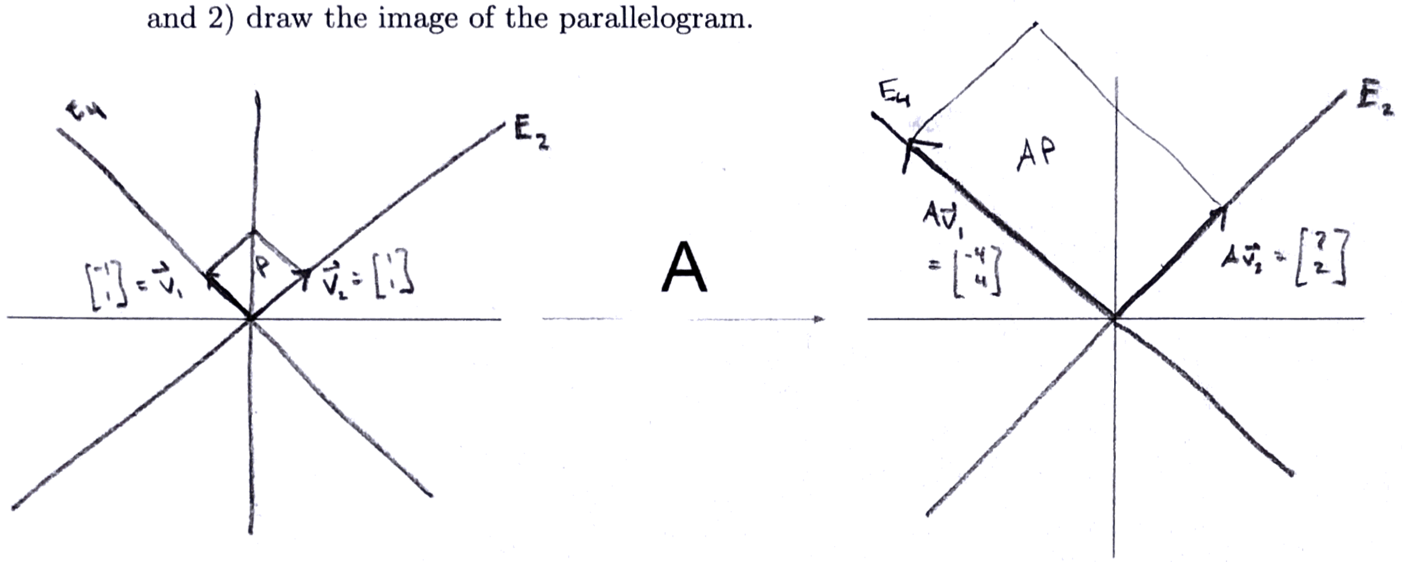
$\lambda = 2$: $E_2 = \text{span}\left\{\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right\}$

(d) Is the matrix diagonalizable? If so, express the matrix in the form UDU^{-1} for some matrices U and D where D is a diagonal matrix. (Compute U^{-1}.) If not, justify.

$U = [\underset{\lambda=4}{\vec{v}_1} \;\; \underset{\lambda=2}{\vec{v}_2}]$ $\quad D = \begin{bmatrix} 4 & 0 \\ 0 & 2 \end{bmatrix}$ $\qquad U = \begin{bmatrix} -1 & 1 \\ 1 & 1 \end{bmatrix} \Rightarrow U^{-1} = \begin{bmatrix} -1/2 & 1/2 \\ 1/2 & 1/2 \end{bmatrix}$

$A = \begin{bmatrix} 3 & -1 \\ -1 & 3 \end{bmatrix} = \begin{bmatrix} -1 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 4 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} -1/2 & 1/2 \\ 1/2 & 1/2 \end{bmatrix}$

(e) This part has many steps. On the graph on the left: 1) draw and label the eigenspaces, 2) add the eigenvectors you chose and label them, and 3) draw a parallelogram with two sides being the eigenvectors. On the graph on the right: 1) draw the eigenspaces, and 2) draw the image of the parallelogram.



(f) Compute the area of the parallelogram.

Area (Parallelogram) $= |\vec{v}_1 \times \vec{v}_2| = |\langle 0, 0, -2\rangle| = 2.$

$$\vec{v}_1 \times \vec{v}_2 = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ -1 & 1 & 0 \\ 1 & 1 & 0 \end{vmatrix} = 0\hat{\imath} + 0\hat{\jmath} + -2\hat{k}$$

(g) Compute the area of the image of the parallelogram.

Area (Parallelogram's Image) $= |A\vec{v}_1 \times A\vec{v}_2| = 16$

$$A\vec{v}_1 \times A\vec{v}_2 = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ -4 & 4 & 0 \\ 2 & 2 & 0 \end{vmatrix} = 0\hat{\imath} + 0\hat{\jmath} + -16\hat{k}$$

(h) How does the area scale? How is that related to the eigenvalues?

The area scales by 8. The length of each side of the parallelogram scales by its respective eigenvalue, so the area scales by the same as the product of the eigenvalues : $2 \cdot 4 = 8$.

(i) Compute the determinant of the matrix D from part (d). How is the determinant related to the eigenvalues?

$\text{Det}(D) = 2 \cdot 4 =$ product of eigenvalues. This also captures how the area is scaling!

(Notice: $\text{Det}(A) = \text{Det}(D)$ also!)

(a) **(BONUS: 6 points)** Recall from M126 that the Taylor series for the exponential function is $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$. We can use this to define the exponential of a square $n \times n$ matrix A:

$$e^A = \sum_{n=0}^{\infty} \frac{A^n}{n!}$$

where we use the convention that $A^0 = I_n$. If possible, use this formula to compute e^A where A is the matrix below.

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

Show all of your steps, and if needed, explain the result.

$$A^2 = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1+1 & 1+1+1 \\ 0 & 1 & 1+1 \\ 0 & 0 & 1 \end{bmatrix}$$

$$A^3 = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1+(1+1) & 1+(1+1)+(1+1+1) \\ 0 & 1 & 1+(1+1) \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 3 & 6 \\ 0 & 1 & 3 \\ 0 & 0 & 1 \end{bmatrix}$$

$$A^4 = \begin{bmatrix} 1 & 3 & 6 \\ 0 & 1 & 3 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1+(1+(1+1)) & 1+(1+(1+1))+(1+(1+1)+(1+1+1)) \\ 0 & 1 & 1+(1+(1+1)) \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 4 & 10 \\ 0 & 1 & 4 \\ 0 & 0 & 1 \end{bmatrix}$$

$$A^n = \begin{bmatrix} 1 & n & \sum_{k=1}^{n} k \\ 0 & 1 & n \\ 0 & 0 & 1 \end{bmatrix}$$

see next page!

$$\text{Then } e^A = \sum_{n=0}^{\infty} \frac{A^n}{n!} = \sum_{n=0}^{\infty} \frac{1}{n!}\begin{bmatrix} 1 & n & \sum_{k=1}^{n} k \\ 0 & 1 & n \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} \sum_{n=0}^{\infty} \frac{1}{n!} & \sum_{n=0}^{\infty} \frac{n}{n!} & (*) \\ 0 & \sum_{n=0}^{\infty} \frac{1}{n!} & \sum_{n=0}^{\infty} \frac{n}{n!} \\ 0 & 0 & \sum_{n=0}^{\infty} \frac{1}{n!} \end{bmatrix} = \begin{bmatrix} e & e & \frac{3}{2}e \\ 0 & e & e \\ 0 & 0 & e \end{bmatrix}$$

Notice: $\sum_{n=0}^{\infty} \frac{1}{n!} = \sum_{n=0}^{\infty} \frac{1^k}{n!} = e^1 = e$ and $\sum_{n=0}^{\infty} \frac{n}{n!} = \sum_{n=1}^{\infty} \frac{1}{(n-1)!} = \sum_{n=0}^{\infty} \frac{1}{n!} = e$

(b) **(BONUS: 2 points)** Recall that in the beginning of class, we defined elementary row operations on a matrix for one purpose: to solve an underlying system of equations. The elementary row operations are exactly the operations that preserve the solution set to the underlying system of equations. What if we had instead defined these operations on columns instead of rows? Then the underlying solutions set would change with each operation, so we would not be able to solve this underlying linear system of equations using "column operations." However, there is a linear system of equations whose solution set is preserved by column operations. What is it?

$$\boxed{A^T \vec{x} = \vec{b}}$$

the system of equations encoded by this matrix equation!

$\underline{n \text{ even}}$: $\sum_{k=1}^{n} k = 1 + \cdots + n = (n+1) \cdot \frac{n}{2} = \frac{1}{2}(n^2+n)$

$\underline{n \text{ odd}}$: $\sum_{k=1}^{n} k = 1 + \cdots + n = (n+1) \cdot \frac{(n-1)}{2} + \left(\frac{n+1}{2}\right) = \frac{1}{2}(n^2+n)$

$(1 + 2 + \underline{\underline{3}} + 4 + 5)$

(0 when $n=0$.)

$$(*) \sum_{n=0}^{\infty} \frac{\sum_{k=1}^{n} k}{n!} = \sum_{n=0}^{\infty} \frac{\frac{1}{2}(n^2+n)}{n!} = \frac{1}{2} \sum_{n=0}^{\infty} \frac{n^2+n}{n!}$$

$$= \frac{1}{2}\left(\sum_{n=0}^{\infty} \frac{n^2}{n!} + \sum_{n=0}^{\infty} \frac{n}{n!}\right)$$

$n=0$ term is 0 in both sums

$$= \frac{1}{2}\left(\sum_{n=1}^{\infty} \frac{n^2}{n!} + \underbrace{\sum_{n=1}^{\infty} \frac{n}{n!}}_{\text{from before!}}\right)$$

$$= \frac{1}{2}\left(\sum_{n=1}^{\infty} \frac{n}{(n-1)!} + e\right)$$

$$= \frac{1}{2}\left(\left(\sum_{n=0}^{\infty} \frac{n+1}{n!}\right) + e\right)$$

$$= \frac{1}{2}\left(\left(\sum_{n=0}^{\infty} \frac{n}{n!} + \sum_{n=0}^{\infty} \frac{1}{n!}\right) + e\right)$$

from before!

$$= \frac{1}{2}\left(e + e + e\right)$$

$$= \frac{3}{2} e$$