MATH 308 F
Quiz 3
July 17, 2020

Name ______________________________

Student ID #______________

HONOR STATEMENT

"I affirm that my work upholds the highest standards of honesty and academic integrity at the University of Washington, and that I have neither given nor received any unauthorized assistance on this exam."

SIGNATURE:______________________________

SOLUTIONS !

1	8	
2	12	
3	5	
Bonus	3	
Total	25	

- Your exam should consist of this cover sheet, followed by 3 problems. Check that you have a complete exam.
- Pace yourself. You have 25 minutes to complete the exam and there are 3 problems. Try not to spend more than 8 minutes on each problem. You will have 10 minutes at the end of the exam to upload your solutions to Gradescope.
- Show all your work and justify your answers.
- Your answers should be exact values rather than decimal approximations. (For example, $\frac{\pi}{4}$ is an exact answer and is preferable to its decimal approximation 0.7854.)
- This is an open book exam, however, you are not allowed to collaborate with anyone.
- There are multiple versions of the exam, you have signed an honor statement, and cheating is a hassle for everyone involved. DO NOT CHEAT.
- Turn your cell phone OFF and put it AWAY for the duration of the exam.

GOOD LUCK!

1. Construct an example. If there is no such example, write NOT POSSIBLE **and justify your answer**. If you provide an example, you **do not** need to justify that your example satisfies the desired conditions, but do show all of your work.

(a) (8 points) **Construct an example** of a basis $\mathscr{B}$ such that for $\vec{x} = \begin{bmatrix}3\\3\end{bmatrix}$, we have $[\vec{x}]_{\mathscr{B}} = \begin{bmatrix}2\\1\end{bmatrix}$ and for $\vec{y} = \begin{bmatrix}0\\3\end{bmatrix}$, we have $[\vec{y}]_{\mathscr{B}} = \begin{bmatrix}1\\2\end{bmatrix}$. (Here, $\vec{x}$ and $\vec{y}$ denote that the vector is written with respect to the standard basis.) How many different choices of a basis $\mathscr{B}$ will satisfy this condition? Why?

Recall, for $\mathscr{B} = \{\vec{u}_1, \vec{u}_2\}$, $U = [\vec{u}_1\ \vec{u}_2]$, we have $U[\vec{x}]_{\mathscr{B}} = \vec{x}$. Let $U = \begin{bmatrix} u_{11} & u_{12} \\ u_{21} & u_{22}\end{bmatrix}$.

① $\begin{bmatrix} u_{11} & u_{12} \\ u_{21} & u_{22}\end{bmatrix}\begin{bmatrix}2\\1\end{bmatrix} = \begin{bmatrix}3\\3\end{bmatrix} \Rightarrow \begin{cases} 2u_{11} + u_{12} = 3 \\ 2u_{21} + u_{22} = 3\end{cases}$

② $\begin{bmatrix} u_{11} & u_{12} \\ u_{21} & u_{22}\end{bmatrix}\begin{bmatrix}1\\2\end{bmatrix} = \begin{bmatrix}0\\3\end{bmatrix} \Rightarrow \begin{cases} u_{11} + 2u_{12} = 0 \\ u_{21} + 2u_{22} = 3\end{cases}$

$\Rightarrow \begin{cases} 2u_{11} + u_{12} = 3 \\ 2u_{21} + u_{22} = 3 \\ u_{11} + 2u_{12} = 0 \\ u_{21} + 2u_{22} = 3 \end{cases}$ (or solve two separate systems)

$$\begin{array}{cccc} u_{11} & u_{12} & u_{21} & u_{22} \end{array}$$
$$\left[\begin{array}{cccc|c} 2 & 1 & 0 & 0 & 3 \\ 0 & 0 & 2 & 1 & 3 \\ 1 & 2 & 0 & 0 & 0 \\ 0 & 0 & 1 & 2 & 3 \end{array}\right] \sim \left[\begin{array}{cccc|c} 1 & 2 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 & 3 \\ 0 & 0 & 1 & 2 & 3 \\ 0 & 0 & 2 & 1 & 3 \end{array}\right] \sim \left[\begin{array}{cccc|c} 1 & 2 & 0 & 0 & 0 \\ 0 & -3 & 0 & 0 & 3 \\ 0 & 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & -3 & -3 \end{array}\right] \Rightarrow \begin{array}{l} u_{22} = 1 \\ u_{21} = 1 \\ u_{12} = -1 \\ u_{11} = 2 \end{array}$$

$\Rightarrow U = \begin{bmatrix} 2 & -1 \\ 1 & 1 \end{bmatrix}$, so we can find $\boxed{\mathscr{B} = \left\{\begin{bmatrix}2\\1\end{bmatrix}, \begin{bmatrix}-1\\1\end{bmatrix}\right\}.}$ This is the <u>only</u> basis that will work since the conditions ① and ② led to a system of equations with exactly one solution.

(b) (**BONUS: 3 points**) Construct an example of a 2×2 matrix whose row space is span $\left\{\begin{bmatrix}3\\4\end{bmatrix}\right\}$ and whose column space is span $\left\{\begin{bmatrix}-1\\3\end{bmatrix}\right\}$.

- If the row space is span$\left\{\begin{bmatrix}3\\4\end{bmatrix}\right\}$, then both row vectors must be multiples of $\begin{bmatrix}3\\4\end{bmatrix}$, so pick two rows: $s\begin{bmatrix}3\\4\end{bmatrix}, t\begin{bmatrix}3\\4\end{bmatrix}$ for some s, t: $\begin{bmatrix} 3s & 4s \\ 3t & 4t \end{bmatrix}$. one of them must be non-zero!
- To pick s, t that work, use the fact that the column vectors must be in the span of $\begin{bmatrix}-1\\3\end{bmatrix}$: $c_1\begin{bmatrix}3s\\3t\end{bmatrix} = \begin{bmatrix}-1\\3\end{bmatrix}$ and $c_2\begin{bmatrix}4s\\4t\end{bmatrix} = \begin{bmatrix}-1\\3\end{bmatrix}$ for some c_1, c_2. So:

① $3sc_1 = -1$
② $3tc_1 = 3$
③ $4sc_2 = -1$
④ $4tc_2 = 3$

Assume $s \neq 0$ (at least one of s,t is non-zero) $\Rightarrow 3c_1 = 4c_2 \Rightarrow c_1 = \frac{4}{3}c_2$ or $c_2 = \frac{3}{4}c_1 \Rightarrow$

① $3sc_1 = -1$
② $3tc_1 = 3$
③ $3sc_1 = -1$
④ $3tc_1 = 3$

redundant

Pick $c_1 = \frac{1}{3}$ (free!), then $s = -1$, $t = 3$ and

(many possible solutions!)

$$\begin{bmatrix} -3 & -4 \\ 9 & 12 \end{bmatrix}.$$

2. Let $A = \begin{bmatrix}\vec{a}_1 & \vec{a}_2 & \vec{a}_3 & \vec{a}_4 & \vec{a}_5\end{bmatrix}$ be a matrix equivalent to the following matrix:

$$\begin{bmatrix} 1 & -3 & 2 & 3 & 5 \\ 0 & 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

(a) (2 points) What is the nullity of A? How do you know?

$\underbrace{\dim(\mathrm{col}(A))}_{= \mathrm{rank}(A)} = 2$ (number of pivot columns)

So by Rank-Nullity: $\mathrm{rank}(A) + \mathrm{nullity}(A) = \underbrace{5}_{\#\text{ of columns}}$

$2 + \mathrm{nullity}(A) = 5$

$\boxed{\mathrm{nullity}(A) = 3}$

(b) (3 points) Generate a basis for the column space of A that includes the vector $\begin{bmatrix} 3 \\ 1 \\ 0 \end{bmatrix}$.

Show all of your steps. If this is not possible, explain why.

This is not possible. By a theorem, row operations preserve dependencies between column vectors, but this tells us nothing about how the column space may or may not change. It is not hard to see that the column space may change under row operations: $\begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$, but $\mathrm{span}\left\{\begin{bmatrix} 1 \\ 2 \end{bmatrix}\right\} \neq \mathrm{span}\left\{\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right\}$.
So, it is impossible to even know whether or not $\begin{bmatrix} 3 \\ 1 \\ 0 \end{bmatrix}$ is in the column space of A.

(c) (7 points) Generate a basis for the row space of A that includes the vector $\begin{bmatrix} 1 \\ -3 \\ 2 \\ 1 \\ -3 \end{bmatrix}$. Show all of your steps. If this is not possible, explain why.

(Many possible solutions!)

Using Method 1, we can say $\mathcal{B}_{\mathrm{row}(A)} = \left\{ \begin{bmatrix} 1 \\ -3 \\ 2 \\ 3 \\ 5 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 4 \end{bmatrix} \right\}$ is a basis for $\mathrm{row}(A)$.

This means $\dim(\mathrm{row}(A)) = 2$.

- Check: is $\begin{bmatrix} 1 \\ -3 \\ 2 \\ 1 \\ -3 \end{bmatrix} \in \mathrm{row}(A)$? If it is, it is a linear combination of the basis elements.

$$\left[\begin{array}{cc|c} 1 & 0 & 1 \\ -3 & 0 & -3 \\ 2 & 0 & 2 \\ 3 & 1 & 1 \\ 5 & 4 & -3 \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & -2 \\ 0 & 4 & -8 \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & -2 \\ 0 & 0 & 0 \end{array}\right] \Rightarrow \begin{array}{l} x_1 = 1 \\ x_2 = -2 \end{array}$$

and $1 \cdot \begin{bmatrix} 1 \\ -3 \\ 2 \\ 3 \\ 5 \end{bmatrix} + -2 \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 4 \end{bmatrix} = \begin{bmatrix} 1 \\ -3 \\ 2 \\ 1 \\ -3 \end{bmatrix}$.

$\Rightarrow \begin{bmatrix} 1 \\ -3 \\ 2 \\ 1 \\ -3 \end{bmatrix}$ is in column space of A.

- By a thm (p.16 in 4.2), since $\dim(\mathrm{row}(A)) = 2$, if we have 2 vectors, we only need to check linear independence.

Let $\mathcal{B} = \left\{ \begin{bmatrix} 1 \\ -3 \\ 2 \\ 1 \\ -3 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 4 \end{bmatrix} \right\}$, notice $\begin{bmatrix} 1 & 0 \\ -3 & 0 \\ 2 & 0 \\ 1 & 1 \\ -3 & 4 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 1 \\ 0 & 4 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$, and since there are pivots in both columns, the vectors are linearly independent.

- By the THM, we can conclude $\mathcal{B}$ is a basis for $\mathrm{row}(A)$.

3. (5 points) Let A be a square matrix such that when you perform the following operations on the rows, you get the identity matrix.

① $R_1 + R_2 \to R_2$

② $3R_3 \to R_3$

$\boxed{R_2 - R_3 \to R_3}$ Not an elementary row operation, but can be broken into elementary row operations (see Practice Quiz #1)

③ $-R_3 \to R_3$

④ $R_2 + R_3 \to R_3$

Compute the determinant of A.

- The first elementary row operation ① does not change the determinant.
- The second ② will scale the determinant by 3.
- The third ③ will scale the determinant by -1.
- The fourth ④ will not change the determinant.

So:

$$\det(A) \cdot 3 \cdot -1 = \det(I_n)$$

$$-3 \det(A) = 1$$

$$\boxed{\det(A) = -\frac{1}{3}}$$