MATH 308 F
Quiz 2
July 10, 2020

Name ______________________________

Student ID #______________

HONOR STATEMENT

"I affirm that my work upholds the highest standards of honesty and academic integrity at the University of Washington, and that I have neither given nor received any unauthorized assistance on this exam."

SIGNATURE:______________________________

SOLUTIONS !

1	8	
2	12	
3	5	
Bonus	3	
Total	25	

- Your exam should consist of this cover sheet, followed by 3 problems. Check that you have a complete exam.
- Pace yourself. You have 25 minutes to complete the exam and there are 3 problems. Try not to spend more than 8 minutes on each problem. You will have 10 minutes at the end of the exam to upload your solutions to Gradescope.
- Show all your work and justify your answers.
- Your answers should be exact values rather than decimal approximations. (For example, $\frac{\pi}{4}$ is an exact answer and is preferable to its decimal approximation 0.7854.)
- This is an open book exam, however, you are not allowed to collaborate with anyone.
- There are multiple versions of the exam, you have signed an honor statement, and cheating is a hassle for everyone involved. DO NOT CHEAT.
- Turn your cell phone OFF and put it AWAY for the duration of the exam.

GOOD LUCK!

1. Construct an example. If there is no such example, write NOT POSSIBLE **and justify your answer**. If you provide an example, you **do not** need to justify that your example satisfies the desired conditions.

(a) (3 points) **Construct an example** of a matrix that first shears the same way the matrix $\begin{bmatrix}1 & 1\\0 & 1\end{bmatrix}$ shears, and then rotates vectors counterclockwise by $\frac{\pi}{4}$, in that order.

shear : $\begin{bmatrix}1 & 1\\0 & 1\end{bmatrix}$ rotate ccw by $\frac{\pi}{4}$: $\begin{bmatrix}\cos(\frac{\pi}{4}) & -\sin(\frac{\pi}{4})\\ \sin(\frac{\pi}{4}) & \cos(\frac{\pi}{4})\end{bmatrix} = \begin{bmatrix}\frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2}\\ \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2}\end{bmatrix}$

$$\underbrace{\begin{bmatrix}\frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2}\\ \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2}\end{bmatrix}\left(\underbrace{\begin{bmatrix}1 & 1\\0 & 1\end{bmatrix}\vec{x}}_{\text{shears } \vec{x}}\right)}_{\text{rotates the sheared } \vec{x}.} \rightsquigarrow \begin{bmatrix}\frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2}\\ \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2}\end{bmatrix}\begin{bmatrix}1 & 1\\0 & 1\end{bmatrix} = \begin{bmatrix}\frac{\sqrt{2}}{2} & 0\\ \frac{\sqrt{2}}{2} & \sqrt{2}\end{bmatrix}$$

(b) (5 points) **Construct an example** a 3×4 matrix A and a 5×3 matrix B such that the linear transformation represented by the matrix BA is one-to-one.

$A_{3\times4}$, $T_A: \mathbb{R}^4 \to \mathbb{R}^3$
$B_{5\times3}$, $T_B: \mathbb{R}^3 \to \mathbb{R}^5$
$\Rightarrow$ $\underbrace{BA}_{5\times4}$ represents $T_B \circ T_A : \mathbb{R}^4 \to \mathbb{R}^5$

- T_A cannot be one-to-one since $4 > 3$. (by a THM). Since $T_A(\vec{x}) = \vec{0}$ has only the trivial solution if and only if T_A is one-to-one, we know that $T_A(\vec{x}) = A\vec{x} = \vec{0}$ has an infinite number of solutions.
- Since T_B is a linear transformation, $T_B(\vec{0}) = \vec{0}$. (or, notice $B\vec{0} = \vec{0}$.)
- This means that for any $\vec{x} \in \ker(T_A)$ $(= \{\vec{x} \in \mathbb{R}^4 : T_A(\vec{x}) = \vec{0}\})$, $T_B \circ T_A(\vec{x}) = T_B(T_A(\vec{x})) = T_B(\vec{0}) = \vec{0}$.
- So, we see that $T_B \circ T_A(\vec{x}) = \vec{0}$ has an infinite number of solutions! So $T_B \circ T_A$ cannot be one-to-one. $\Rightarrow$ $\boxed{\text{NOT POSSIBLE}}$

2. (Part a and part b are unrelated.) You must show **all** of your steps to receive credit in each part ~~to recieve credit.~~

   (a) (5 points) Solve the following matrix equation for X. You may assume all matrices are square and invertible. You must show all of your work to receive credit.

$$XA - BXA + CXA = D$$

$$(X - BX)A + CXA = D$$

$$((X - BX) + (X))A = D$$

$$(X - BX + CX)\underbrace{AA^{-1}}_{I_n} = DA^{-1}$$

$$X - BX + CX = DA^{-1}$$

since $X = I_n X$

$$I_n X - BX + CX = DA^{-1}$$

$$(I_n - B)X + CX = DA^{-1}$$

$$((I_n - B) + C)X = DA^{-1}$$

$$(I_n - B + C)^{-1}(I_n - B + C)X = (I_n - B + C)^{-1}DA^{-1}$$

$$\boxed{X = (I_n - B + C)^{-1}DA^{-1}}$$

   (b) (7 points) Let

$$A = \begin{bmatrix} 1 & 1 & 3 & 12 \\ -2 & 70 & 0 & 10 \\ 1 & 2 & -1 & 4 \\ 2 & 15 & 1 & 2020 \end{bmatrix}, \quad \vec{b} = \begin{bmatrix} 4 \\ -2 \\ 0 \\ 3 \end{bmatrix}$$

It turns out that $\vec{b}$ is a linear combination of the first and third column vectors in A. Compute $A^{-1}\vec{b}$. You may assume A is invertible. (No credit will be awarded for explicitly computing A^{-1}. In fact, I am reasonably confident you cannot compute it in less than 25 minutes without using a calculator. There is a way to solve this without explicitly computing the inverse.)

We use the fact that $\vec{b}$ is a linear combination of the first and third column vectors:

$$x_1\begin{bmatrix} 1 \\ -2 \\ 1 \\ 2 \end{bmatrix} + x_3\begin{bmatrix} 3 \\ 0 \\ -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 4 \\ -2 \\ 0 \\ 3 \end{bmatrix} \rightsquigarrow \left[\begin{array}{cc|c} 1 & 3 & 4 \\ -2 & 0 & -2 \\ 1 & -1 & 0 \\ 2 & 1 & 3 \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & 3 & 4 \\ 0 & 6 & 6 \\ 0 & -4 & -4 \\ 0 & -5 & -5 \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & 3 & 4 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{array}\right] \Rightarrow \begin{array}{l} x_3 = 1 \\ x_1 = 1 \end{array}$$

(or by inspection!)

This means: $$1\cdot\begin{bmatrix} 1 \\ -2 \\ 1 \\ 2 \end{bmatrix} + 0\cdot\begin{bmatrix} 1 \\ 70 \\ 2 \\ 15 \end{bmatrix} + 1\cdot\begin{bmatrix} 3 \\ 0 \\ -1 \\ 1 \end{bmatrix} + 0\cdot\begin{bmatrix} 12 \\ 10 \\ 4 \\ 2020 \end{bmatrix} = \begin{bmatrix} 4 \\ -2 \\ 0 \\ 3 \end{bmatrix}$$

i.e.: $$\underbrace{\begin{bmatrix} 1 & 1 & 3 & 12 \\ -2 & 70 & 0 & 10 \\ 1 & 2 & -1 & 4 \\ 2 & 15 & 1 & 2020 \end{bmatrix}}_{A}\underbrace{\begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}}_{\vec{x}} = \underbrace{\begin{bmatrix} 4 \\ -2 \\ 0 \\ 3 \end{bmatrix}}_{\vec{b}}$$

Since A is invertible:

$$A\vec{x} = \vec{b} \iff A^{-1}\vec{b} = \vec{x}$$

$$\text{so } A^{-1}\vec{b} = \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}.$$

3. (5 points) Determine whether or not the following sets are subspaces. Justify your answer.

(a) $S = \left\{ \vec{v} = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} \in \mathbb{R}^3 : v_1 + v_2 + v_3 = 0 \right\}$

Method 1: Check Conditions

① Is $\vec{0} \in S$?

$\vec{0} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ and $0+0+0=0$, so $\vec{0} \in S$.

② If $\vec{u}, \vec{v} \in S$, is $\vec{u}+\vec{v} \in S$?

If $\vec{u}, \vec{v} \in S$, then $\vec{u} = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix}$ and $u_1+u_2+u_3=0$

$\vec{v} = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}$ and $v_1+v_2+v_3=0$

Then $\vec{u}+\vec{v} = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} + \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} u_1+v_1 \\ u_2+v_2 \\ u_3+v_3 \end{bmatrix}$ and

$(u_1+v_1)+(u_2+v_2)+(u_3+v_3) = \underbrace{u_1+u_2+u_3}_{0} + \underbrace{v_1+v_2+v_3}_{0} = 0$.

So $\vec{u}+\vec{v} \in S$.

③ If $\vec{u} \in S$, r scalar, is $r\vec{u} \in S$?

$\vec{u} \in S$, then $\vec{u} = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix}$ where $u_1+u_2+u_3=0$.

$r\vec{u} = r\begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} = \begin{bmatrix} ru_1 \\ ru_2 \\ ru_3 \end{bmatrix}$ and

$ru_1 + ru_2 + ru_3 = r\underbrace{(u_1+u_2+u_3)}_{0} = 0$.

So $r\vec{u} \in S$.

Method 2: Exhibit S as the null space of a matrix.

$S = \left\{ \vec{v} = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} \in \mathbb{R}^3 : v_1+v_2+v_3 = 0 \right\}$

$S = \left\{ \vec{v} = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} \in \mathbb{R}^3 : \begin{bmatrix} 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \end{bmatrix} \right\}$

so $S = \{ \vec{v} \in \mathbb{R}^3 : \begin{bmatrix} 1 & 1 & 1 \end{bmatrix} \vec{v} = \vec{0} \}$.

So S is the null space of the 1×3 matrix $\begin{bmatrix} 1 & 1 & 1 \end{bmatrix}$. By a theorem, since null spaces are subspaces, S is a subspace.

(b) (**BONUS: 3 points**) $S = \{\vec{v} \in \mathbb{R}^3 : C\vec{v} + B\vec{v} = B^2\vec{v}\}$, where B and C are both 3×3 matrices, but neither is necessarily invertible.

You can check the conditions, but here's the easier way:

$S = \{ \vec{v} \in \mathbb{R}^3 : C\vec{v} + B\vec{v} = B^2\vec{v} \}$

$S = \{ \vec{v} \in \mathbb{R}^3 : C\vec{v} + B\vec{v} - B^2\vec{v} = \vec{0} \}$

$S = \{ \vec{v} \in \mathbb{R}^3 : (C + B - B^2)\vec{v} = \vec{0} \}$

OR THINK:

$C\vec{v} + B\vec{v} = B^2\vec{v}$

$C\vec{v} + B\vec{v} - B^2\vec{v} = \vec{0}$

$(C+B-B^2)\vec{v} = \vec{0}$

if you don't like the set notation!

So S is the nullspace of the matrix $(C+B-B^2)$, thus by a theorem, S is a subspace.