MATH 308 F

Quiz 1

July 6, 2020

Name ______________________________

Student ID #______________

HONOR STATEMENT

"I affirm that my work upholds the highest standards of honesty and academic integrity at the University of Washington, and that I have neither given nor received any unauthorized assistance on this exam."

SIGNATURE:______________________________

SOLUTIONS!

1	5	
2	10	
3	10	
Total	25	

- Your exam should consist of this cover sheet, followed by 3 problems. Check that you have a complete exam.
- Pace yourself. You have 25 minutes to complete the exam and there are 3 problems. Try not to spend more than 8 minutes on each problem. You will have 10 minutes at the end of the exam to upload your solutions to Gradescope.
- Show all your work and justify your answers.
- Your answers should be exact values rather than decimal approximations. (For example, $\frac{\pi}{4}$ is an exact answer and is preferable to its decimal approximation 0.7854.)
- This is an open book exam, however, you are not allowed to collaborate with anyone.
- There are multiple versions of the exam, you have signed an honor statement, and cheating is a hassle for everyone involved. DO NOT CHEAT.
- Turn your cell phone OFF and put it AWAY for the duration of the exam.

GOOD LUCK!

1. (5 Points) Construct an example. If there is no such example, write NOT POSSIBLE. You **do not** need to justify that your example satisfies the desired conditions, nor do you need to justify answers to the additional questions.

   (a) **Construct an example** of a linear transformation $T : \mathbb{R}^m \to \mathbb{R}^n$ that is both onto and one-to-one. Are there any constraints on what m and n can be?

   Let $T(\vec{x}) = A\vec{x}$ where $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$.

   NOTE: pivot in every column and every row!

   For any example, $\boxed{m = n}$.

   (b) **Construct an example** of a set of vectors $\{\vec{u}_1, \vec{u}_2, \vec{u}_3\}$ such that span $\{\vec{u}_1, \vec{u}_2, \vec{u}_3\} = \mathbb{R}^2$, but span $\{\vec{u}_1, \vec{u}_2\} \neq \mathbb{R}^2$.

   Let $\vec{u}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \vec{u}_2 = \begin{bmatrix} 2 \\ 0 \end{bmatrix}, \vec{u}_3 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$: $\left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\}$.

2. (10 Points) Determine if one of the vectors is in the span of the other vectors. Show all of your work and justify your procedure.

$$\vec{u} = \begin{bmatrix} 4 \\ -1 \\ 3 \end{bmatrix}, \vec{v} = \begin{bmatrix} 3 \\ 5 \\ -2 \end{bmatrix}, \vec{w} = \begin{bmatrix} -5 \\ 7 \\ -7 \end{bmatrix}$$

We will use the theorem that says a set of vectors is linearly dependent if and only if one of the vectors in the set is in the span of the others.

We first check for linear dependence: $A\vec{x} = \vec{0}$ has how many solutions?

$$\left[\begin{array}{ccc|c} 4 & 3 & -5 & 0 \\ -1 & 5 & 7 & 0 \\ 3 & -2 & -7 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} -1 & 5 & 7 & 0 \\ 4 & 3 & -5 & 0 \\ 3 & -2 & -7 & 0 \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|c} -1 & 5 & 7 & 0 \\ 0 & 23 & 23 & 0 \\ 0 & 13 & 14 & 0 \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|c} -1 & 5 & 7 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 13 & 14 & 0 \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|c} -1 & 5 & 7 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right] \rightarrow \begin{array}{l} x_3 = 0 \\ x_2 = 0 \\ x_1 = 0 \end{array} \rightarrow \text{we only have one solution, the trivial solution!}$$

$\{\vec{u}, \vec{v}, \vec{w}\}$

This means the set is linearly independent (by definition). So by the theorem stated above, this means none of the vectors can be in the span of the others.

3. (10 points) Find the conditions on a and b under which the following set of vectors **does not** span $\mathbb{R}^3$. Show all of your work and justify your procedure.

$$\left\{ \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} a \\ b \\ 3 \end{bmatrix} \right\}$$

By the unifying theorem, since we have $\underline{\underline{3}}$ vectors in $\mathbb{R}^3$, the set does not span $\mathbb{R}^3$ <u>if and only if</u> the set is a linearly dependent set. So, we want a and b such that

$$\left[\begin{array}{ccc|c} 1 & -1 & a & 0 \\ 2 & 0 & b & 0 \\ 3 & 1 & 3 & 0 \end{array}\right]$$

has infinitely many solutions:

$$\left[\begin{array}{ccc|c} 1 & -1 & a & 0 \\ 2 & 0 & b & 0 \\ 3 & 1 & 3 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & -1 & a & 0 \\ 0 & 2 & b-2a & 0 \\ 0 & 4 & 3-3a & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & -1 & a & 0 \\ 0 & 2 & b-2a & 0 \\ 0 & 0 & 3-2b+a & 0 \end{array}\right].$$

To have infinitely many solutions, we need a free variable, which means we need a column without a pivot. The first two columns have a pivot, which means we need to pick a and b so that the third column does not have a pivot. (This means we need a zero row!) We can pick a, b to satisfy:

$$\boxed{3 - 2b + a = 0}$$

third column won't have a pivot!

Then, the system has an infinite number of solutions, so the set of vectors is linearly dependent, and by the unifying theorem we can conclude this is the constraint on a and b to ensure the set <u>does</u> <u>not</u> span $\mathbb{R}^3$.