MATH 308 F
Practice Quiz
June 26, 2020

Name ______________________________

Student ID #________________

HONOR STATEMENT

"I affirm that my work upholds the highest standards of honesty and academic integrity at the University of Washington, and that I have neither given nor received any unauthorized assistance on this exam."

SIGNATURE:______________________________

SOLUTIONS!

1	8	
2	7	
3	10	
Total	25	

- Your exam should consist of this cover sheet, followed by 3 problems. Check that you have a complete exam.
- Pace yourself. You have 25 minutes to complete the exam and there are 3 problems. Try not to spend more than 10 minutes on each problem. You will have 10 minutes at the end of the exam to upload your solutions to Gradescope.
- Show all your work and justify your answers.
- Your answers should be exact values rather than decimal approximations. (For example, $\frac{\pi}{4}$ is an exact answer and is preferable to its decimal approximation 0.7854.)
- This is an open book exam, however, you are not allowed to collaborate with anyone.
- There are multiple versions of the exam, you have signed an honor statement, and cheating is a hassle for everyone involved. DO NOT CHEAT.
- Turn your cell phone OFF and put it AWAY for the duration of the exam.

GOOD LUCK!

1. (8 Points) True / False and Short Answer

Clearly indicate whether the statement is true or false. **If true, justify your answer. If false, provide a counterexample or give justification.**

(a) **TRUE / FALSE** A system of equations with more variables than equations always has at least one solution.

False :

1 pt.

$$\begin{cases} x_1 + x_2 + x_3 = 0 \\ x_1 + x_2 + x_3 = 1 \end{cases}$$

← parallel planes! (no intersection)

1 pt.

Construct an example. If there is no such example, write NOT POSSIBLE. You **do not** need to justify that your example satisfies the desired conditions, nor do you need to justify answers to the additional questions.

(b) **Give an example** of a linear system of equations with exactly three variables with no solutions such that when you remove an equation (you can pick which one), the remaining system of equations has exactly one solution.

3pts (no partial!) →

$$\begin{cases} x_1 + x_2 + x_3 = 0 \\ x_2 + x_3 = 0 \\ x_3 = 0 \\ x_1 + x_2 + x_3 = 1 \end{cases}$$

← remove, what's left is a triangular system! (so it has exactly one solution.)

Short Answer Question.

(c) One of the following is **not** an elementary row operation. Identify it, and write it as a sequence of elementary row operations. Assume the operations are being applied to a matrix with 3 rows.

Edit

i. $\frac{1}{2}R_2 \to R2$: Scale the second row by $\frac{1}{2}$.

ii. $R_3 + (-1)R_1 \to R1$: Add the third row to negative one times the first row and put the result into row 1.

iii. $R_3 + (-1)R_1 \to R3$: Add the third row to negative one times the first row and put the result into row 3.

ii) is not an elementary row operation. We can write it as two elementary row operations in sequence:

1 pt.

2 pts

First : $(-1)R_1 \to R_1$: (scale row 1 by -1)

Then : $R_3 + R_1 \to R_1$: (add row 3 to row 1 and put the result into Row 1.)

2. (7 Points) Are the following matrices equivalent under row operations? **Fully justify your answer** using theorems or techniques from the class. **Show all of your work** (list all elementary row operations that you use if you use them as a part of your argument).

$$\begin{bmatrix} 1 & 1 & 2 \\ 0 & 1 & 0 \\ 1 & 4 & 2 \end{bmatrix}, \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

4 pts

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \underset{\substack{-R_3+R_2 \to R_2 \\ -R_3+R_1 \to R_1}}{\sim} \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \underset{-R_2+R_1 \to R_1}{\sim} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$ ← reduced row echelon form.

← labeling required!

$$\begin{bmatrix} 1 & 1 & 2 \\ 0 & 1 & 0 \\ 1 & 4 & 2 \end{bmatrix} \underset{-R_1+R_3 \to R_3}{\sim} \begin{bmatrix} 1 & 1 & 2 \\ 0 & 1 & 0 \\ 0 & 3 & 0 \end{bmatrix} \underset{-3R_2+R_3 \to R_3}{\sim} \begin{bmatrix} 1 & 1 & 2 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \underset{-R_2+R_1 \to R_1}{\sim} \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$ ← reduced row echelon form.

3 pts

Now, recall that for a given matrix, the reduced row echelon form matrix is <u>unique</u>, so since the equivalent reduced row echelon form matrices are not the same: $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \neq \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$, the two matrices are <u>not</u> equivalent.

<u>Alternatively</u> (Using the fact that equivalent matrices give rise to systems of equations with the same underlying solution set.)

3 pts

Let $A\vec{x}=\vec{0}$ and $B\vec{x}=\vec{0}$ be systems of equations, where $A = \begin{bmatrix} 1 & 1 & 2 \\ 0 & 1 & 0 \\ 1 & 4 & 2 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$. Since $[A \mid \vec{0}] \sim \left[\begin{array}{ccc|c} 1 & 0 & 2 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$ ← free variable! missing a pivot! and $[B \mid \vec{0}] \sim \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right]$, we can see that $A\vec{x}=\vec{0}$ has infinite solutions, but $B\vec{x}=\vec{0}$ has one solution ($x_1=x_2=x_3=0$), so the two systems have <u>different</u> solution sets, hences A cannot be equivalent to B.

Here are a few examples of Justification for #2 that would not get credit:

1. The first matrix doesn't have a pivot in the third column. ↙ 1 pt.

   (You would be totally right! <u>But</u>, you need to connect this idea - not having a pivot - to the meaning of "equivalent matrices." Notice how the alternative justification does this!)

2. I cannot use row operations to turn one of them into the other. ↙ 0 pts

   (Also, totally correct! But, you need to justify why you can't.)

3. One of the matrices is equivalent to a matrix with a row of zeroes, so it cannot be equivalent to the other. ↙ 1 pt.

   (Also true! But this needs justification. There are an infinite number of row operations you could do... why is it that <u>none</u> of them will work?)

3. (10 points) Recall that quadratic polynomials are polynomials of the form $y = a_0+a_1x+a_2x^2$, where a_0, a_1 and a_2 are any real numbers.

(a) For what values of c are there infinitely many different quadratic polynomials such that $(2,0)$, $(5,0)$, $(c,0)$ are points on the quadratic polynomial?

plug into form of polynomial

$(2,0) \rightarrow 0 = a_0 + 2a_1 + 4a_2$

$(5,0) \rightarrow 0 = a_0 + 5a_1 + 25a_2$

$(c,0) \rightarrow 0 = a_0 + ca_1 + c^2a_2$

$\Rightarrow \begin{cases} a_0 + 2a_1 + 4a_2 = 0 \\ a_0 + 5a_1 + 25a_2 = 0 \\ a_0 + ca_1 + c^2a_2 = 0 \end{cases}$

solve sys. of equations ... when do we have ∞'ly many solutions?

6pts

$$\begin{array}{ccc} (a_0) & (a_1) & (a_2) \end{array}$$
$$\left[\begin{array}{ccc|c} 1 & 2 & 4 & 0 \\ 1 & 5 & 25 & 0 \\ 1 & c & c^2 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 2 & 4 & 0 \\ 0 & 3 & 21 & 0 \\ 0 & c-2 & c^2-4 & 0 \end{array}\right]$$

← want a free variable, so a column without a pivot. This means we need a zero row.

so: since we have already picked a pivot in row 1, we can focus on row 2 and row 3:

need $KR_2 + R_3 \rightarrow R_3$ = zero row

so:
$$\begin{array}{rcccc|c} & KR_2 = & 0 & 3K & 21K & 0 \\ + & R_3 = & 0 & c-2 & c^2-4 & 0 \\ \hline & & 0 & 0 & 0 & 0 \end{array}$$

$\Rightarrow$ system of equations

$\begin{cases} 3K + (c-2) = 0 \\ 21K + (c^2-4) = 0 \end{cases}$ solve for K $\Rightarrow K = \frac{-(c-2)}{3}$, plug in

$21\left(\frac{-(c-2)}{3}\right) + (c^2-4) = 0$

$c^2 - 7c + 10 = 0$

$(c-2)(c-5) = 0$

$\Rightarrow \boxed{c = 2 \text{ or } 5}$

(b) Using the same points as in part (a), for what values of c does there exist exactly one polynomial passing through the points? It turns out to be the same polynomial for all of these values. What is that polynomial? justify that it is the same for all of these values.

4pts

There are ∞'ly many solutions for $c=2$ or $c=5$. This means if $c \neq 2, 5$, then the system of equations above has either 0 or 1 solutions. Since the system is homogeneous, it always has at least one solution (the trivial solution where $a_0=0, a_1=0, a_2=0$). Thus, for $c \neq 2, 5$, the system will have exactly one solution, and it will always be the trivial solution.

We can compute what the solution means in terms of the polynomial:

$y = a_0 + a_1x + a_2x^2$

$y = 0 + 0x + 0 \cdot x^2$

$\boxed{y = 0}$