MATH 308 M

Exam II

February 21, 2020

Name ______________________________

Student ID #____________________

HONOR STATEMENT

"I affirm that my work upholds the highest standards of honesty and academic integrity at the University of Washington, and that I have neither given nor received any unauthorized assistance on this exam."

SIGNATURE:______________________________

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|---|---|---|
| 1 | 16 | |
| 2 | 6 | |
| 3 | 14 | |
| 4 | 14 | |
| Bonus | 8 | |
| Total | 50 | |

- Your exam should consist of this cover sheet, followed by 4 problems and a bonus question. Check that you have a complete exam.
- Pace yourself. You have 50 minutes to complete the exam and there are 4 problems. Try not to spend more than 10 minutes on each problem.
- Show all your work and justify your answers.
- Your answers should be exact values rather than decimal approximations. (For example, $\frac{\pi}{4}$ is an exact answer and is preferable to its decimal approximation 0.7854.)
- You may use TI-30X IIS calculator and one 8.5×11-inch sheet of handwritten notes. All other electronic devices (including graphing calculators) are forbidden.
- The use of headphones or earbuds during the exam is not permitted.
- There are multiple versions of the exam, you have signed an honor statement, and cheating is a hassle for everyone involved. DO NOT CHEAT.
- Turn your cell phone OFF and put it AWAY for the duration of the exam.

GOOD LUCK!

1. (16 Points) True / False and Short Answer.

   Clearly indicate whether the statement is true or false and **justify your answer**.

   (a) **TRUE** / **FALSE** $\mathbb{R}^2$ is a subspace of $\mathbb{R}^4$.

   (b) **TRUE** / **FALSE** Let $A$ be an $n \times m$ matrix such that $A^T\vec{x} = \vec{b}$ is a consistent linear system for every $\vec{b}$ in $\mathbb{R}^m$. Then $n < m$.

Give an example of each of the following. If there is no such example, write NOT POSSIBLE and **justify why it is not possible.** If you provide an example, you do not need to justify why the example works.

(c) **Give an example** of a matrix $A$ such that $A^8 = I_2$, but $A^k \neq I_2$ for an integer $k$ such that $0 < k < 8$.

(d) **Give an example** of a linear transformation $T : \mathbb{R}^3 \mapsto \mathbb{R}^3$ such that $\text{Range}(T) = \text{Ker}(T)$. [*Hint:* Consider the Rank-Nullity Theorem.]

(e) **Give an example** of a linear transformation $T : \mathbb{R}^m \mapsto \mathbb{R}^n$ where $m < n$ and $T$ is not one-to-one.

2. (6 points)

   (a) Recall from M126 that the Taylor series for the exponential function is $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$. Since we know how to take powers of a matrix, we can use this to define the exponential of a square $n \times n$ matrix A:

   $$e^A = \sum_{n=0}^{\infty} \frac{1}{n!} A^n$$

   where we use the convention that $A^0 = I_n$ and $0! = 1$. Compute $e^A$ where $A$ is the matrix $\begin{bmatrix} 2 & -1 \\ 4 & -2 \end{bmatrix}$. **Hint:** If you compute the powers of the matrix correctly, you will not need to take an infinite sum.

   (b) (**BONUS: 3 points**) Using the same instructions from part (a), compute $e^A$ where $A$ is the matrix $\begin{bmatrix} 7 & 0 \\ 0 & 9 \end{bmatrix}$.

3. (14 Points) Let $T$ be a linear transformation such that $T(\vec{x}) = A\vec{x}$ where $A$ and its equivalent reduced row echelon form is given by

$$A = \begin{bmatrix} 0 & 2 & 6 & 14 \\ 2 & -1 & 1 & 11 \\ 1 & -1 & -1 & 2 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 2 & 9 \\ 0 & 1 & 3 & 7 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

(a) What is the domain of $T$?

(b) What is the codomain of T?

(c) Is the null space of $A$ in the domain or codomain? Give a basis for Null($A$).

(d) Is the column space of $A$ in the domain or codomain? Give a basis for Col($A$).

(e) Is the row space of $A$ in the domain or codomain? Give a basis for Row($A$).

(f) Is $T$ one-to-one? Onto? **Justify your answer.**

4. (14 points)

   (a) Let $A$ be any square matrix $A$. Show that the set $S$ consisting of the vectors $\vec{v}$ that are fixed by the matrix A is a subspace, i.e. show that the set of vectors $\vec{v}$ such that $A\vec{v} = \vec{v}$ is a subspace.

   (b) Now, let

$$A = \begin{bmatrix} 3 & -2 & 3 \\ 4 & -3 & 6 \\ 2 & -2 & 4 \end{bmatrix}.$$

   Give a basis for $S$ as given in part (a). You may use the back of this page if you need more space.

**Bonus:** (5 points) A quadratic form is a function $Q(\vec{x})$ that satisfies the following two properties:

(a) $Q(r\vec{x}) = r^2 Q(\vec{x})$ for any vector $\vec{x}$ and $r$ any real number (scalar).

(b) Fix a vector $\vec{y}$. Then $T(\vec{x}) = Q(\vec{x}+\vec{y}) - Q(\vec{x}) - Q(\vec{y})$ is a degree one polynomial in the entries of $\vec{x}$ (i.e. T is a linear equation).

Show that for a $2 \times 2$ matrix $A$, the function $Q : \mathbb{R}^2 \to \mathbb{R}$ given by

$$Q(\vec{x}) = \vec{x}^T A\vec{x}$$

is a quadratic form. (Note: $\vec{x}^T$ is a row vector.)