MATH 308 N
Exam II
November 13, 2019

Name ______________________________

Student ID #________________

HONOR STATEMENT

"I affirm that my work upholds the highest standards of honesty and academic integrity at the University of Washington, and that I have neither given nor received any unauthorized assistance on this exam."

SIGNATURE:______________________________

SOLUTIONS!

1	8	
2	14	
3	14	
4	14	
Bonus	5	
Total	50	

- Your exam should consist of this cover sheet, followed by 4 problems and a bonus question. Check that you have a complete exam.
- Pace yourself. You have 50 minutes to complete the exam and there are 4 problems. Try not to spend more than 12 minutes on each problem.
- Show all your work and justify your answers.
- Your answers should be exact values rather than decimal approximations. (For example, $\frac{\pi}{4}$ is an exact answer and is preferable to its decimal approximation 0.7854.)
- You may use TI-30X IIS calculator and one 8.5×11-inch sheet of handwritten notes. All other electronic devices (including graphing calculators) are forbidden.
- The use of headphones or earbuds during the exam is not permitted.
- There are multiple versions of the exam, you have signed an honor statement, and cheating is a hassle for everyone involved. DO NOT CHEAT.
- Turn your cell phone OFF and put it AWAY for the duration of the exam.

GOOD LUCK!

1. (8 Points) True / False and Short Answer.

Clearly indicate whether the statement is true or false and **justify your answer** using only material we have covered in the course.

(a) **TRUE** / **FALSE** If A and B are both invertible $n \times n$ matrices, then AB is invertible.

Let T_A, T_B be linear transformations represented by A, B, resp. Since T_A and T_B are onto, $T_A \circ T_B$ is onto. Similarly, since T_A and T_B are one-to-one, $T_A \circ T_B$ is one-to-one. Thus, $T_A \circ T_B$ is invertible, and so AB is invertible.

OR

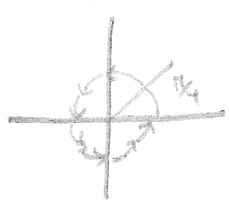
$(AB)^{-1} = B^{-1}A^{-1}$. Since A^{-1} and B^{-1} exists, $(AB)^{-1}$ exists.

(b) **TRUE** / **FALSE** $\mathbb{R}^2$ is a subspace of $\mathbb{R}^3$.

$\mathbb{R}^2$ is not a subset of $\mathbb{R}^3$

Give an example of each of the following. If there is no such example, write NOT POSSIBLE and **justify your answer**.

(c) **Give an example** of a matrix A such that $A^8 = I_2$, but $A^k \neq I_2$ for an integer k such that $0 < k < 8$.



rotate by $\frac{\pi}{4}$: $\begin{bmatrix} \cos(\frac{\pi}{4}) & -\sin(\frac{\pi}{4}) \\ \sin(\frac{\pi}{4}) & \cos(\frac{\pi}{4}) \end{bmatrix} = \begin{bmatrix} \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \end{bmatrix}$.

(d) **Give an example** of a linear transformation $T : \mathbb{R}^3 \mapsto \mathbb{R}^4$, where $T(\vec{x}) = A\vec{x}$ for some matrix A, such that $\text{Rank}(A) = 2$ and $\text{Nullity}(A) = 2$.

Not possible by rank-nullity theorem: $2 + 2 = 4 \neq 3$.

$\boxed{\text{NOT POSSIBLE}}$

2. (14 Points) Let T be a linear transformation such that $T(\vec{x}) = A\vec{x}$ where A and its equivalent reduced row echelon form is given by

$$A = \begin{bmatrix} 1 & 3 & 21 & 22 \\ -1 & 1 & 3 & -2 \\ 1 & 1 & 9 & 12 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 3 & 7 \\ 0 & 1 & 6 & 5 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

(a) What is the codomain of T?

$\mathbb{R}^3$

(b) Is the null space of A in the domain or codomain? Give a basis for Null(A).

Null space of A is in the domain, $\mathbb{R}^4$.

$$A \sim \begin{bmatrix} 1 & 0 & 3 & 7 \\ 0 & 1 & 6 & 5 \\ 0 & 0 & 0 & 0 \end{bmatrix} \Rightarrow \text{Let } x_4 = t,\ x_3 = s. \quad \text{Then } x_2 = -6s - 5t,\ x_1 = -3s - 7t$$

$$\vec{x} = \begin{bmatrix} -3s - 7t \\ -6s - 5t \\ s \\ t \end{bmatrix} = s\begin{bmatrix} -3 \\ -6 \\ 1 \\ 0 \end{bmatrix} + t\begin{bmatrix} -7 \\ -5 \\ 0 \\ 1 \end{bmatrix}$$

these vectors are always linearly independent (in hom. solution)

$$\text{null}(A) = \text{span}\left\{ \begin{bmatrix} -3 \\ -6 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -7 \\ -5 \\ 0 \\ 1 \end{bmatrix} \right\}. \qquad \mathcal{B}_{\text{Null}(A)} = \left\{ \begin{bmatrix} -3 \\ -6 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -7 \\ -5 \\ 0 \\ 1 \end{bmatrix} \right\},$$

(c) Is the column space of A in the domain or codomain? Give a basis for Col(A).

Codomain.

$$\mathcal{B}_{\text{Col}(A)} = \left\{ \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}, \begin{bmatrix} 3 \\ 1 \\ 1 \end{bmatrix} \right\}.$$

(by Method 2)

(d) Is the row space of A in the domain or codomain? Give a basis for Row(A).

Domain.

$$\mathcal{B}_{\text{row}(A)} = \left\{ \begin{bmatrix} 1 \\ 0 \\ 3 \\ 7 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 6 \\ 5 \end{bmatrix} \right\}.$$

(By Method 1).

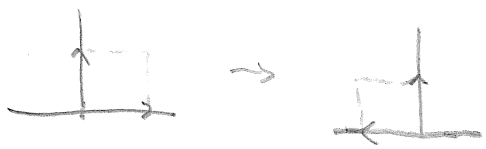
(e) Is T one-to-one? Onto? **Justify your answer.**

Not one-to-one: $\ker(T) = \text{null}(A) \neq \{\vec{0}\}$.

Not onto: $\text{Range}(T) = \text{col}(A) \neq \mathbb{R}^3$,

3. (14 points)

(a) Produce a 2×2 matrix that reflects $\mathbb{R}^2$ over the y-axis.



$$\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$$

(b) Produce a 2×2 matrix that rotates $\mathbb{R}^2$ by 270 degrees ($\frac{3\pi}{2}$ radians) counter-clockwise.

$$\begin{bmatrix} \cos(\frac{3\pi}{2}) & -\sin(\frac{3\pi}{2}) \\ \sin(\frac{3\pi}{2}) & \cos(\frac{3\pi}{2}) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$$

(c) Compute the matrix that represents a reflection of $\mathbb{R}^2$ over the y-axis then a rotation by 270 degrees counter-clockwise, in that order. Call this matrix C.

$$\mathrm{Rot}(\mathrm{Ref}(\vec{x})) = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}\vec{x}$$

reflect

then rotate

$$\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} = \boxed{\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = C}$$

(d) Compute the matrix that represents a rotation of $\mathbb{R}^2$ by 270 degrees counter-clockwise, then a reflection over the y-axis, in that order. Call this matrix D.

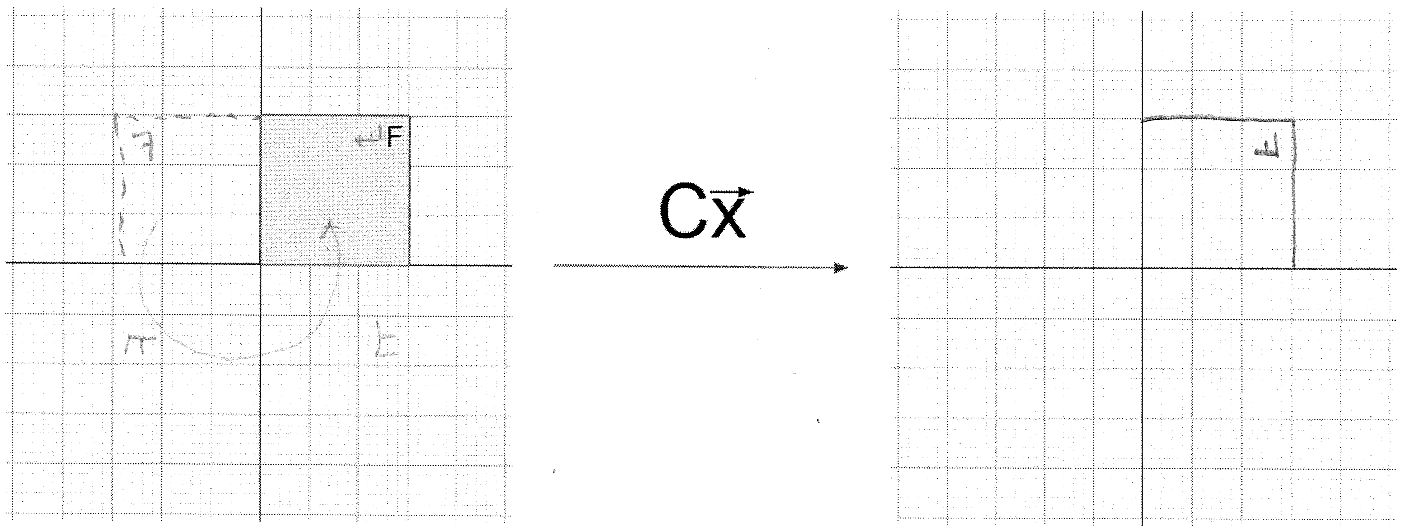
$$\mathrm{Ref}(\mathrm{Rot}(\vec{x})) = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}\vec{x}$$

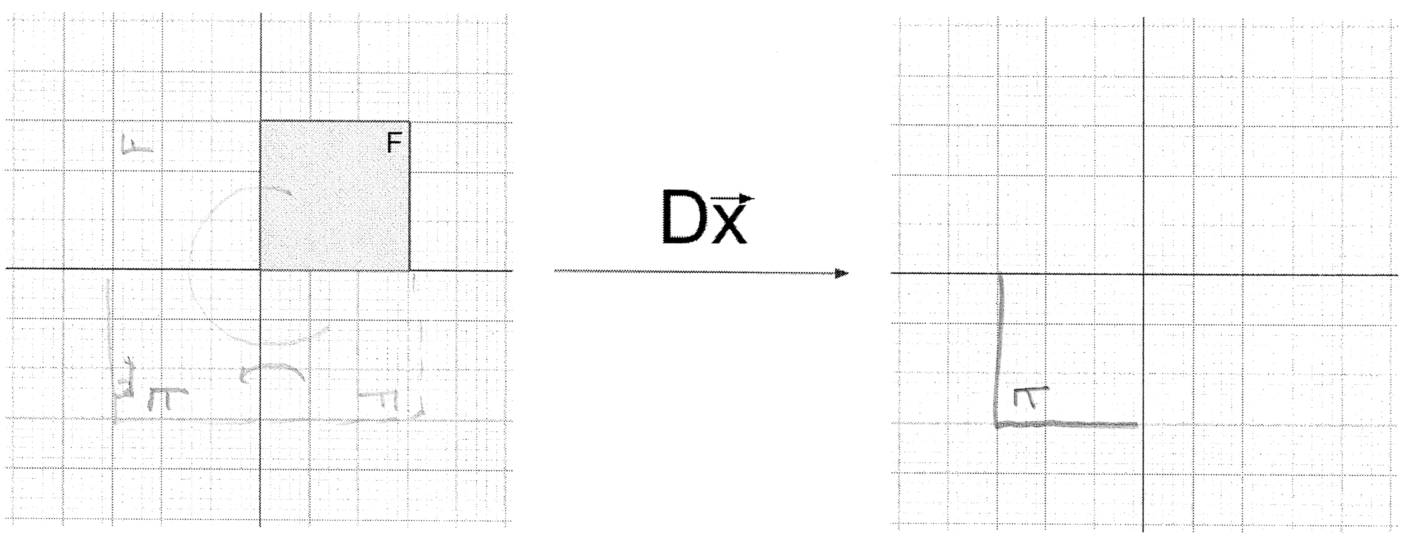
rotate

then reflect

$$\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} = \boxed{\begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix} = D}$$

(e) Complete the following drawings. Show where the unit square gets mapped and draw F with the correct orientation on the new square. Matrix C denotes the matrix from part c, and matrix D denotes the matrix from part d.





(f) **Fill in the blank.** This is a geometric proof of the fact that matrix multiplication is not commutative.

4. (14 points)

(a) Let A be any square matrix A and λ any scalar. Show that the set S consisting of the vectors $\vec{v}$ such that $A\vec{v} = \lambda\vec{v}$ is a subspace.

Method 1: $A\vec{v} = \lambda\vec{v}$

$A\vec{v} - \lambda\vec{v} = \vec{0}$

$(A - \lambda I)\vec{v} = \vec{0}$

$\vec{v}$ satisfying the above equation is a subspace since Null $(A - \lambda I)$ is a subspace.

Method 2:

- $\vec{0} \in S$?

  $A\vec{0} = \vec{0} = \lambda \cdot \vec{0}$. ✓

- If $\vec{u}, \vec{v} \in S$, is $\vec{u} + \vec{v} \in S$?

  $A(\vec{u}+\vec{v}) = A\vec{u} + A\vec{v} = \lambda\vec{u} + \lambda\vec{v} = \lambda(\vec{u}+\vec{v})$, ✓

- If $\vec{u} \in S$, r scalar, is $r\vec{u} \in S$?

  $A(r\vec{u}) = rA\vec{u} = r\lambda\vec{u} = \lambda(r\vec{u})$ ✓

S is a subspace.

(b) Now, let $\lambda = -2$ and let

$$A = \begin{bmatrix} 1 & -3 & 3 \\ 3 & -5 & 3 \\ 6 & -6 & 4 \end{bmatrix}.$$

Give a basis for S as given in part (a). You may use the back of this page if you need more space.

Method 1: $A\vec{v} = -2\vec{v}$

$A\vec{v} + 2\vec{v} = \vec{0}$

$(A + 2I_3)\vec{v} = 0$

$$A + 2I_3 = \begin{bmatrix} 3 & -3 & 3 \\ 3 & -3 & 3 \\ 6 & -6 & 6 \end{bmatrix} \sim \begin{bmatrix} 1 & -1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

So, $$\left[\begin{array}{ccc|c} 3 & -3 & 3 & 0 \\ 3 & -3 & 3 & 0 \\ 6 & -6 & 6 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

lin. ind (as before!)

$x_3 = t$

$x_2 = s$

$x_1 = s - t$

$$\Rightarrow \vec{v} = \begin{bmatrix} s-t \\ s \\ t \end{bmatrix} = s\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + t\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} \Rightarrow \mathcal{B}_{\text{null}(A)} = \left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} \right\}.$$

Method 2: ("Follow your nose")

Let $\vec{v} = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}$. Find all $\vec{v}$ that satisfy: $A\vec{v} = -2\vec{v}. = \begin{bmatrix} -2v_1 \\ -2v_2 \\ -2v_3 \end{bmatrix}$

$$\left[\begin{array}{ccc|c} 1 & -3 & 3 & -2v_1 \\ 3 & -5 & 3 & -2v_2 \\ 6 & -6 & 4 & -2v_3 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & -3 & 3 & -2v_1 \\ 0 & 4 & -6 & -2v_2+6v_1 \\ 0 & 12 & -14 & -2v_3+12v_1 \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|c} 1 & -3 & 3 & -2v_1 \\ 0 & 4 & -6 & -2v_2+6v_1 \\ 0 & 0 & 4 & -2v_3+12v_1+6v_2-18v_1 \end{array}\right]$$

underlying system of equations:

$$\begin{aligned} v_1 - 3v_2 + 3v_3 &= -2v_1 \\ 4v_2 - 6v_3 &= -2v_2 + 6v_1 \\ 4v_3 &= -2v_3 + 6v_2 - 6v_1 \end{aligned}$$

Oof... well, keep following your nose

Move all terms to one side.

$$\begin{aligned} 3v_1 - 3v_2 + 3v_3 &= 0 \\ -6v_1 + 6v_2 - 6v_3 &= 0 \\ 6v_1 - 6v_2 + 6v_3 &= 0 \end{aligned}$$

... hmm. (compare to method 1)

$$\left[\begin{array}{ccc|c} 3 & -3 & 3 & 0 \\ -6 & 6 & -6 & 0 \\ 6 & -6 & 6 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Let $v_3 = t$

$v_2 = s$

Then, $v_1 = s - t$

linearly independent!

The set S of vectors $\vec{v}$ satisfying the equation $\Rightarrow$ $\vec{v} = \begin{bmatrix} s-t \\ s \\ t \end{bmatrix} = s\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + t\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$ $\Rightarrow \mathcal{B}_S = \left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \right\}$.

BONUS: (5 points) Short answer. To receive credit, you must fully justify your reasoning.

(a) Define a bracket operation on two matrices X and Y by $[X, Y] = XY - YX$. Now, let A and B be two matrices such that $(A + B)^2 = A^2 + 2AB + B^2$. Compute $[A, B]$.

$(A+B)^2 = A^2 + AB + BA + B^2$. $A^2 + AB + BA + B^2 = A^2 + 2AB + B^2$ only if $\underline{AB + BA = 2AB}$, i.e. $\underline{BA = AB}$.

$[A, B] = AB - BA = AB - AB = 0$.

(b) Now, let C and D be matrices such that $[C, D] = A$, where A is not the zero-matrix. What can we say about C and D?

$[C, D] = CD - DC = A$

$CD = DC + A \Rightarrow CD \neq DC$,

so we can conclude

C and D <u>do not commute</u>.

(c) Summarize what this bracket operation is doing in terms of the algebraic structure we have developed for matrices.

The bracket operation identifies when two matrices commute, or "tells us how far from commuting" two matrices are.