MATH 308 M
Exam I
January 31, 2020

Name ______________________________

Student ID #______________

HONOR STATEMENT

"I affirm that my work upholds the highest standards of honesty and academic integrity at the University of Washington, and that I have neither given nor received any unauthorized assistance on this exam."

SIGNATURE:______________________________

1	16	
2	10	
3	10	
4	14	
Bonus	5	
Total	50	

SOLUTIONS!

- Your exam should consist of this cover sheet, followed by 4 problems and a bonus question. Check that you have a complete exam.
- Pace yourself. You have 60 minutes to complete the exam and there are 4 problems. Try not to spend more than 12 minutes on each problem.
- Show all your work and justify your answers.
- Your answers should be exact values rather than decimal approximations. (For example, $\frac{\pi}{4}$ is an exact answer and is preferable to its decimal approximation 0.7854.)
- You may use TI-30X IIS calculator and one 8.5×11-inch sheet of handwritten notes. All other electronic devices (including graphing calculators) are forbidden.
- The use of headphones or earbuds during the exam is not permitted.
- There are multiple versions of the exam, you have signed an honor statement, and cheating is a hassle for everyone involved. DO NOT CHEAT.
- Turn your cell phone OFF and put it AWAY for the duration of the exam.

GOOD LUCK!

1. (16 Points) True / False and Short Answer (2 pages).

Clearly indicate whether the statement is true or false. **If true, justify your answer. If false, provide a counterexample.**

(a) TRUE / **FALSE** If $\{\vec{u}_1, \vec{u}_2, \vec{u}_3\}$ is a linearly dependent set of vectors in $\mathbb{R}^5$, then $\{\vec{u}_1, \vec{u}_2, \vec{u}_3\, \vec{u}_4\}$ is linearly dependent for any choice of $\vec{u}_4$ in $\mathbb{R}^5$.

True. If $\{\vec{u}_1, \vec{u}_2, \vec{u}_3\}$ is linearly dependent, then there is a non-trivial solution to $x_1\vec{u}_1 + x_2\vec{u}_2 + x_3\vec{u}_3 = \vec{0}$. Let that solution be $x_1 = c_1$, $x_2 = c_2$, $x_3 = c_3$. (Then at least one of the c_i's is not zero and $c_1\vec{u}_1 + c_2\vec{u}_2 + c_3\vec{u}_3 = \vec{0}$.) Then

$$c_1\vec{u}_1 + c_2\vec{u}_2 + c_3\vec{u}_3 + 0\vec{u}_4 = \vec{0}$$

is a non-trivial solution for any choice of $\vec{u}_4$, thus $\{\vec{u}_1, \vec{u}_2, \vec{u}_3, \vec{u}_4\}$ is linearly dependent.

(b) **TRUE** / FALSE Let $\{\vec{v_1}, \vec{v_2}, \vec{v_3}, \vec{v_4}\}$ be a set of linearly dependent vectors in $\mathbb{R}^3$. Then $\vec{v_1} \in \text{span}\{\vec{v_2}, \vec{v_3}, \vec{v_4}\}$.

False: Let $\vec{v}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$, $\vec{v}_2 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$, $\vec{v}_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$, $\vec{v}_4 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$. ← something like this was all you were required to say.

Then $\{\vec{v}_1, \vec{v}_2, \vec{v}_3, \vec{v}_4\}$ is linearly dependent ($\vec{v}_2 = \vec{v}_3 + \vec{v}_4$, and by THM, $\vec{v}_2 \in \text{span}\{\vec{v}_1, \vec{v}_3, \vec{v}_4\}$ implies linear dependence). <u>But</u>,

$$\vec{v}_1 \notin \text{span}\{\vec{v}_2, \vec{v}_3, \vec{v}_4\}.$$

Give an example of each of the following. If there is no such example, write NOT POSSIBLE. You **do not** need to justify that your example satisfies the desired conditions.

(c) **Give an example** of a linear system of equations consisting of an infinite number of distinct equations and a finite number of variables with precisely one solution.

$$\{y = mx\}_{m \text{ any real number}}.$$

(d) **Give an example** of a system of equations with more variables than equations that has no solution.

$$\begin{cases} x_1 + x_2 + x_3 = 0 \\ x_1 + x_2 + x_3 = 2 \end{cases}$$

(parallel planes)

any (announced during exam)

(e) **Give an example** of a system of equations with no solution, but when one equation is removed, the new system has infinite solutions.

$$\begin{cases} x_1 + x_2 = 0 \\ x_1 + x_2 = 2 \end{cases}$$

← no solution (parallel lines)
but remove either, and you have infinite solutions.

(f) **Give an example** of a matrix in echelon form with a pivot in every row where there are more columns than rows.

pivot in every row →

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}_{3\times 4}$$

2. (10 Points)

(a) Clearly circle the sets of vectors below that permit a linear combination equal to $\begin{bmatrix} a \\ b \end{bmatrix}$ for any a, b real numbers. You are not required to show work on part (a). You may use your geometric intuition.

$$\left\{ \begin{bmatrix} 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 3 \end{bmatrix} \right\}, \left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 5 \\ 5 \end{bmatrix} \right\}, \left\{ \begin{bmatrix} 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 10 \\ 0 \end{bmatrix} \right\},$$

$$\left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ -1 \end{bmatrix} \right\}, \left\{ \begin{bmatrix} 3 \\ 5 \end{bmatrix}, \begin{bmatrix} -6 \\ -10 \end{bmatrix} \right\}, \left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right\},$$

(b) Assume you circled the correct sets in part (a). What is the span of each of the circled sets?

$\mathbb{R}^2$

(c) Which of the circled sets are linearly independent sets? Which are not? **Justify your answer.**

linearly independent: $\left\{ \begin{bmatrix} 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 3 \end{bmatrix} \right\}$ and $\left\{ \begin{bmatrix} 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 10 \\ 0 \end{bmatrix} \right\}$ by the unifying theorem since we have 2 vectors that span $\mathbb{R}^2$.

linearly dependent: $\left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ -1 \end{bmatrix} \right\}$ since we have 3 vectors in $\mathbb{R}^2$, and $3 > 2$ (apply theorem!)

OR: $1 \cdot \begin{bmatrix} 0 \\ -1 \end{bmatrix} + 1 \cdot \begin{bmatrix} 0 \\ 1 \end{bmatrix} + 0 \cdot \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$, non-trivial solution.

(d) Which of the non-circled sets are linearly independent sets? Which are not? **Justify your answer.**

linearly independent: $\left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right\}$ since $x_1 \begin{bmatrix} 1 \\ 0 \end{bmatrix} \left(= \begin{bmatrix} x_1 \\ 0 \end{bmatrix} \right) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ has only the trivial solution $x_1 = 0$.

linearly dependent: $\left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 5 \\ 5 \end{bmatrix} \right\}$ since $\begin{bmatrix} 5 \\ 5 \end{bmatrix} \in \text{span}\left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\}$, by a theorem.
$\left(\begin{bmatrix} 5 \\ 5 \end{bmatrix} = 5 \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right)$

$\left\{ \begin{bmatrix} 3 \\ 5 \end{bmatrix}, \begin{bmatrix} -6 \\ -10 \end{bmatrix} \right\}$ since $\begin{bmatrix} -6 \\ -10 \end{bmatrix} \in \text{span}\left\{ \begin{bmatrix} 3 \\ 5 \end{bmatrix} \right\}$, by a theorem.
$\left(\begin{bmatrix} -6 \\ -10 \end{bmatrix} = -2 \begin{bmatrix} 3 \\ 5 \end{bmatrix} \right)$

3. (10 points) Let $S = \{\vec{u}_1, \vec{u}_2, \vec{u}_3, \vec{u}_4\}$ be a set of vectors in $\mathbb{R}^4$ where

$$\vec{u}_1 = \begin{bmatrix} 3 \\ 2 \\ -1 \\ 1 \end{bmatrix}, \vec{u}_2 = \begin{bmatrix} -5 \\ 4 \\ 2 \\ 0 \end{bmatrix}, \vec{u}_3 = \begin{bmatrix} 1 \\ 3 \\ 2 \\ 0 \end{bmatrix}, \vec{u}_4 = \begin{bmatrix} 1 \\ 8 \\ 0 \\ 2 \end{bmatrix}$$

(a) It turns out S is a linearly dependent set. Show this.

$R_2 \leftrightarrow R_3$, then $\frac{1}{2}R_2 \to R_2$

$$\left[\begin{array}{cccc|c} 3 & -5 & 1 & 1 & 0 \\ 2 & 4 & 3 & 8 & 0 \\ -1 & 2 & 2 & 0 & 0 \\ 1 & 0 & 0 & 2 & 0 \end{array}\right] \sim \left[\begin{array}{cccc|c} 1 & 0 & 0 & 2 & 0 \\ 2 & 4 & 3 & 8 & 0 \\ -1 & 2 & 2 & 0 & 0 \\ 3 & -5 & 1 & 1 & 0 \end{array}\right] \sim \left[\begin{array}{cccc|c} 1 & 0 & 0 & 2 & 0 \\ 0 & 4 & 3 & 4 & 0 \\ 0 & 2 & 2 & 2 & 0 \\ 0 & -5 & 1 & -5 & 0 \end{array}\right] \sim \left[\begin{array}{cccc|c} 1 & 0 & 0 & 2 & 0 \\ 0 & 1 & 1 & 1 & 0 \\ 0 & 4 & 3 & 4 & 0 \\ 0 & -5 & 1 & -5 & 0 \end{array}\right]$$

$$\sim \left[\begin{array}{cccc|c} 1 & 0 & 0 & 2 & 0 \\ 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 6 & 0 & 0 \end{array}\right] \sim \left[\begin{array}{cccc|c} 1 & 0 & 0 & 2 & 0 \\ 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] \rightarrow \text{the solution: } \begin{array}{l} x_4 = t \\ x_3 = 0 \\ x_2 = -t \\ x_1 = -2t \end{array}$$

free variable! So linearly dependent.

solution in vector form: $\vec{x} = \begin{bmatrix} -2t \\ -t \\ 0 \\ t \end{bmatrix} = t \begin{bmatrix} -2 \\ -1 \\ 0 \\ 1 \end{bmatrix}.$

this is telling you x_1, x_2, x_3, x_4 so that $x_1\vec{u}_1 + x_2\vec{u}_2 + x_3\vec{u}_3 + x_4\vec{u}_4 = \vec{0}$

(b) Now write one of the vectors in the set S as a linear combination of the remaining vectors.

Find a non-trivial solution: let $t = 1$, then $\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} -2 \\ -1 \\ 0 \\ 1 \end{bmatrix}$

is a solution to

$$x_1\vec{u}_1 + x_2\vec{u}_2 + x_3\vec{u}_3 + x_4\vec{u}_4 = \vec{0}$$

so

$$-2\vec{u}_1 + -\vec{u}_2 + 0\vec{u}_3 + 1\cdot\vec{u}_4 = \vec{0}.$$

Solve for one of the vectors!

$$\boxed{\vec{u}_4 = 2\vec{u}_1 + \vec{u}_2}$$

Check: (it works!)

$$2\begin{bmatrix} 3 \\ 2 \\ -1 \\ 1 \end{bmatrix} + 1\cdot\begin{bmatrix} -5 \\ 4 \\ 2 \\ 0 \end{bmatrix} = \begin{bmatrix} 6-5 \\ 4+4 \\ -2+2 \\ 2+0 \end{bmatrix} = \begin{bmatrix} 1 \\ 8 \\ 0 \\ 2 \end{bmatrix} \checkmark$$

4. (14 points)

(a) Find all values of a such that the set of vectors $\{\vec{v_1}, \vec{v_2}, \vec{v_3}\}$ are linearly dependent. Then find all values of a such that the set spans $\mathbb{R}^3$. If you use a shortcut, justify your work with a theorem.

$$\vec{v_1} = \begin{bmatrix} 2 \\ a \\ 1 \end{bmatrix}, \vec{v_2} = \begin{bmatrix} 0 \\ 1 \\ a \end{bmatrix}, \vec{v_3} = \begin{bmatrix} 1 \\ -1 \\ 3 \end{bmatrix}.$$

Find a so that we have nontrivial solutions to: $x_1\vec{v}_1 + x_2\vec{v}_2 + x_3\vec{v}_3 = \vec{0}$.

$$\left[\begin{array}{ccc|c} 2 & 0 & 1 & 0 \\ a & 1 & -1 & 0 \\ 1 & a & 3 & 0 \end{array}\right] \sim$$

... so, if you do it this way, you have to be very careful! If you multiply a row by $\frac{1}{a}$ or (a^2-1), you have to be sure you are not "dividing by 0, or multiplying a row by zero. You may have to check values of a independently!

(much) easier way: Change the order of the vectors! (Still just looking for a linear combination of the vectors ... order doesn't matter here.)

$$\left[\begin{array}{ccc|c} 1 & 2 & 0 & 0 \\ -1 & a & 1 & 0 \\ 3 & 1 & a & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 2 & 0 & 0 \\ 0 & a+2 & 1 & 0 \\ 0 & -5 & a & 0 \end{array}\right]$$

← now that we have a pivot here, we can consider these two rows independently.

Want nontrivial solutions, which means we need a free variable. The only way that will happen here is if we have a zero row.

this means we need $c \cdot R_2 = R_3$ for some scalar constant c, so we can subtract: $R_3 + -cR_2 = 0$-row:

so: $\begin{cases} (a+2)c = -5 \\ 1 \cdot c = a \end{cases}$ plug in $\rightarrow (a+2)a = -5$

$a^2+2a+5=0$

and $a_{1,2} = \frac{-2 \pm \sqrt{4-20}}{2}$,

no real solutions

So **no** a will make the set linearly dependent!

(note, if $a=0$

$$\left[\begin{array}{ccc|c} 1 & 2 & 0 & 0 \\ 0 & 2 & 1 & 0 \\ 0 & -5 & 0 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 2 & 0 & 0 \\ 0 & 2 & 1 & 0 \\ 0 & 0 & -\frac{5}{2} & 0 \end{array}\right]$$

← not an a we are looking for!)

Next, notice that this means all a make the set linearly independent. By the Unifying theorem, all a make the set span $\mathbb{R}^3$!

BONUS: (5 points) The span of the following vectors is a plane.

$$\begin{bmatrix}1\\0\\2\end{bmatrix}, \begin{bmatrix}2\\1\\-5\end{bmatrix}, \begin{bmatrix}3\\2\\-12\end{bmatrix}$$

Write the scalar equation of the plane.

$$\left[\begin{array}{ccc|c} 1 & 2 & 3 & b_1 \\ 0 & 1 & 2 & b_2 \\ 2 & -5 & -12 & b_3 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 2 & 3 & b_1 \\ 0 & 1 & 2 & b_2 \\ 0 & -9 & -18 & b_3 - 2b_1 \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|c} 1 & 2 & 3 & b_1 \\ 0 & 1 & 2 & b_2 \\ 0 & 0 & 0 & b_3 - 2b_1 + 9b_2 \end{array}\right]$$

Need b_1, b_2, b_3 to satisfy: $-2b_1 + 9b_2 + b_3 = 0$.

All $\vec{b}$ in the span must satisfy this equation! and $\vec{0}$ is in the span, so the span of the vectors is

$$\boxed{-2b_1 + 9b_2 + b_3 = 0}$$