MATH 308 N
Exam I
October 18, 2019

Name ______________________________

Student ID #______________

HONOR STATEMENT

"I affirm that my work upholds the highest standards of honesty and academic integrity at the University of Washington, and that I have neither given nor received any unauthorized assistance on this exam."

SIGNATURE:______________________________

SOLUTIONS!

1	15	
2	10	
3	10	
4	15	
Bonus	5	
Total	50	

- Your exam should consist of this cover sheet, followed by 4 problems and a bonus question. Check that you have a complete exam.
- Pace yourself. You have 50 minutes to complete the exam and there are 4 problems. Try not to spend more than 10 minutes on each problem.
- Show all your work and justify your answers.
- Your answers should be exact values rather than decimal approximations. (For example, $\frac{\pi}{4}$ is an exact answer and is preferable to its decimal approximation 0.7854.)
- You may use TI-30X IIS calculator and one 8.5×11-inch sheet of handwritten notes. All other electronic devices (including graphing calculators) are forbidden.
- The use of headphones or earbuds during the exam is not permitted.
- There are multiple versions of the exam, you have signed an honor statement, and cheating is a hassle for everyone involved. DO NOT CHEAT.
- Turn your cell phone OFF and put it AWAY for the duration of the exam.

GOOD LUCK!

1. (15 Points) Short Answer.

   Give an example of each of the following. If there is no such example, write NOT POSSIBLE.

   (a) **Give an example** of three linearly dependent vectors in $\mathbb{R}^3$ that span $\mathbb{R}^3$.

   NOT POSSIBLE

   (b) **Give an example** of a linear system of equations consisting of an infinite number of distinct equations with precisely one solution.

   $\{ y = mx ; \ m \text{ is any scalar} \}$.

   (c) **Give an example** of three distinct vectors in $\mathbb{R}^2$ that do not span $\mathbb{R}^2$.

   $\begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 3 \\ 0 \end{bmatrix}$

   (d) **Give an example** of a matrix in echelon form with a pivot in every column where there are more columns than rows.

   NOT POSSIBLE

   (e) **Give an example** of a set of linearly dependent vectors in $\mathbb{R}^3$ such that when you remove **any one** of the three vectors, the remaining set is linearly independent and spans $\mathbb{R}^3$.

   NOT POSSIBLE

2. (10 points) Let $x^2 + y^2 = ax + by + c$ be the formula for a circle in $\mathbb{R}^2$ that contains the points $(2,0)$, $(2,4)$, $(4,2)$. Find the values for a, b and c using techniques we learned in this class.

Plug in points to get 3 equations involving a, b, and c:

$(2,0) \Rightarrow 2^2 + 0^2 = 2a + 0y + 1 \cdot c$

$4 = 2a + 0y + 1 \cdot c$

$(2,4) \Rightarrow 2^2 + 4^2 = 2a + 4b + 1 \cdot c$

$20 = 2a + 4b + 1 \cdot c$

$(4,2) \Rightarrow 4^2 + 2^2 = 4a + 2b + 1 \cdot c$

$20 = 4a + 2b + 1 \cdot c$

System of equations

$$\begin{cases} 2a + c = 4 \\ 2a + 4b + c = 20 \\ 4a + 2b + c = 20 \end{cases}$$

solve system:

$$\left[\begin{array}{ccc|c} 2 & 0 & 1 & 4 \\ 2 & 4 & 1 & 20 \\ 4 & 2 & 1 & 20 \end{array}\right] \sim \left[\begin{array}{ccc|c} 2 & 0 & 1 & 4 \\ 0 & 4 & 0 & 16 \\ 0 & 2 & -1 & 12 \end{array}\right] \sim \left[\begin{array}{ccc|c} 2 & 0 & 1 & 4 \\ 0 & 4 & 0 & 16 \\ 0 & 0 & -1 & 4 \end{array}\right]$$

$c = -4$

$4b = 16 \Rightarrow b = 4$

$2a + c = 4 \Rightarrow 2a - 4 = 4 \Rightarrow 2a = 8 \Rightarrow a = 4$

$$\Rightarrow \begin{cases} a = 4 \\ b = 4 \\ c = -4 \end{cases}.$$

3. (10 points) Determine if one of the vectors is in the span of the other vectors.
HINT: Can you think of a relationship between span and linear independence that might help you determine this without having to explicitly write one of the vectors as a linear combination of the others? Apply a theorem from class.

$$\vec{u} = \begin{bmatrix}1\\7\\8\\4\end{bmatrix}, \vec{v} = \begin{bmatrix}-1\\3\\5\\2\end{bmatrix}, \vec{w} = \begin{bmatrix}3\\1\\-2\\0\end{bmatrix}$$

THM A set of vectors is linearly dependent if and only if one of the vectors is in the span of the others.

Check to see if the set is linearly dependent. If so, then one of the vectors is in the span of the others. If not, then none of the vectors is in the span of the other vectors.

checking for linear dependence:

$$\left[\begin{array}{ccc|c} 1 & -1 & 3 & 0 \\ 7 & 3 & 1 & 0 \\ 8 & 5 & -2 & 0 \\ 4 & 2 & 0 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & -1 & 3 & 0 \\ 0 & 10 & -20 & 0 \\ 0 & 13 & -26 & 0 \\ 0 & 6 & -12 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & -1 & 3 & 0 \\ 0 & 10 & -20 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

free variable.

We have an infinite number of solutions, thus $\{\vec{u}, \vec{v}, \vec{w}\}$ is a linearly dependent set. Applying the theorem, we see that one of the vectors is in the span of the others.

4. (15 points)

(a) Find all values of a such that the set of vectors $\{\vec{v_1}, \vec{v_2}, \vec{v_3}\}$ is **linearly dependent**.

$$\vec{v_1} = \begin{bmatrix} 1 \\ 2 \\ -3 \end{bmatrix}, \vec{v_2} = \begin{bmatrix} 1 \\ -10 \\ a \end{bmatrix}, \vec{v_3} = \begin{bmatrix} 2 \\ a \\ -5 \end{bmatrix}.$$

Want infinitely many solutions:

$$\left[\begin{array}{ccc|c} 1 & 1 & 2 & 0 \\ 2 & -10 & a & 0 \\ -3 & a & -5 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 1 & 2 & 0 \\ 0 & -12 & a-4 & 0 \\ 0 & a+3 & 1 & 0 \end{array}\right]$$

← To have an infinite number of solutions, need a zero row so that we have a free variable.

This means some scalar c times R_2 needs to be negative R_3 ∴ $cR_2 + R_3 \to \underline{0\text{-row}}$.

In other words:

$$\begin{cases} c \cdot (-12) + (a+3) = 0 \\ c \cdot (a-4) + 1 = 0 \end{cases}$$

← solve equation 1 for c, plug into equation 2 and solve for a.

$-12c = -(a+3)$

$c = \frac{a+3}{12}$

$\frac{a+3}{12}(a-4) + 1 = 0$

$\frac{(a+3)(a-4) + 12}{12} = 0$.

need numerator $= 0$, so: $(a+3)(a-4) + 12 = 0$

$a^2 + 3a - 4a - 12 + 12 = 0$

$a^2 - a = 0$

$a(a-1) = 0 \Rightarrow \boxed{a = 0, 1}$

(b) Can you use part (a) to give the values of a such that the set spans $\mathbb{R}^3$? If so, what are they? Explain your thinking.

Yes. In $\mathbb{R}^3$, if three vectors span $\mathbb{R}^3$, then by the unifying theorem, this is equivalent to the property that the three vectors are linearly independent. In other words, the three vectors span $\mathbb{R}^3$ if and only if they are linearly independent. The vectors above are linearly independent when they are not linearly dependent, thus, picking any a other than $a=0$ or $a=1$ will work.

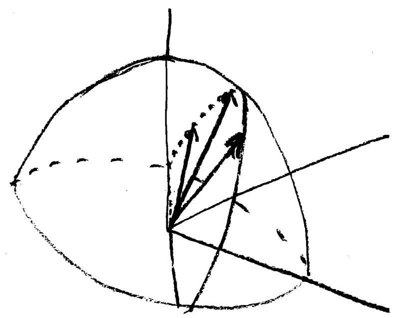
BONUS: (5 points) Consider the upper-half hemisphere,

$$H = \{(x, y, z) \mid x^2 + y^2 + z^2 = 1 \text{ and } z > 0\}.$$

Let $\vec{v_1}, \vec{v_2}, \vec{v_3}$ be vectors starting at the origin, each pointing to a **unique** point on H.

(a) Is the set $\{\vec{v_1}, \vec{v_2}, \vec{v_3}\}$ necessarily linearly independent? Explain your answer.

No, it is possible that one of the vectors is in the span of the others. Geometrically, it is possible that all three vectors are spanning the same plane:



(b) Remove **one** vector (any one of them) from the set. Is the remaining set linearly independent? Explain your answer.

Yes! Since the vectors are unique, and both magnitude 1, the only way they can be linearly dependent (i.e. one is in the span of the other ... i.e. they both point along the same line!) is if one is the negative of the other:

$$\vec{v}_i = -\vec{v}_j.$$

But, this cannot happen if both vectors have positive z-entry.

(c) Repeat part (b), but modify the set H: let $H = \{(x, y, z) \mid x^2 + y^2 + z^2 = 1 \text{ and } \boxed{z \geq 0}\}$.

Now, we can have a set where two of the vectors are negatives of each other. For example, $\vec{v}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$, $\vec{v}_2 = \begin{bmatrix} -1 \\ 0 \\ 0 \end{bmatrix}$, and $\vec{v}_3$ could be any other vector. So no.