Lecture #15

6.2 Diagonalization

Throughout the course, you may have noticed that diagonal matrices have nice properties and are, generally speaking, easier to work with than general matrices. Here are a couple examples illustrating why diagonal matrices are "nicer."

① Matrix Powers

REM If A is diagonal, A^k is easy to compute!

$$A = \begin{bmatrix} a_{11} & 0 & 0 \\ 0 & a_{22} & 0 \\ 0 & 0 & a_{33} \end{bmatrix}, \quad A^k = \begin{bmatrix} a_{11}^k & 0 & 0 \\ 0 & a_{22}^k & 0 \\ 0 & 0 & a_{33}^k \end{bmatrix}$$

(This is in the 3.2 notes.)

② Matrix multiplication

If A and B are both $n \times n$ diagonal matrices...

then A and B will commute! I.e. $AB = BA$.

e.g. $A = \begin{bmatrix} a_{11} & 0 & 0 \\ 0 & a_{22} & 0 \\ 0 & 0 & a_{33} \end{bmatrix}, \quad B = \begin{bmatrix} b_{11} & 0 & 0 \\ 0 & b_{22} & 0 \\ 0 & 0 & b_{33} \end{bmatrix}$

then, $AB = \begin{bmatrix} a_{11}b_{11} & 0 & 0 \\ 0 & a_{22}b_{22} & 0 \\ 0 & 0 & a_{33}b_{33} \end{bmatrix} = BA.$

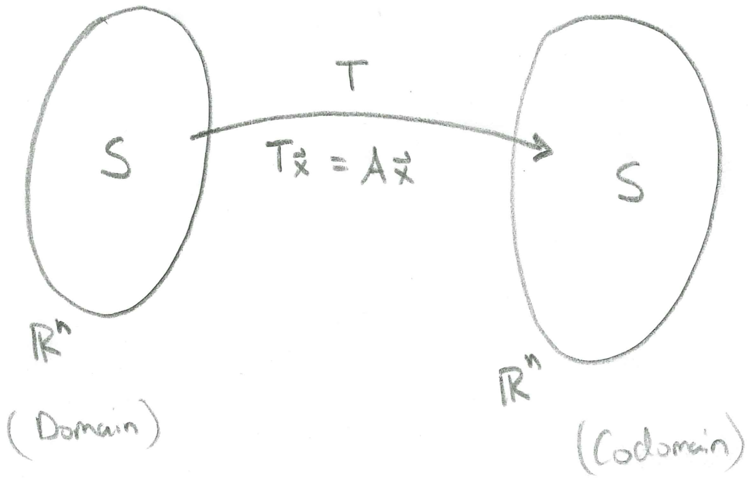
These nice properties may lead us to ask the following question... is it possible to "diagonalize" a matrix that is not diagonal? This certainly seems like wishful thinking, right? A matrix is a matrix, and you can't really change it... right? Well... wrong. We (sort of) can.

Recall how we tend to use the following language when talking about linear transformations. "For any linear transformation, we can find a matrix that represents the linear transformation." And we go on to say things like "let A be a matrix that represents the linear transformation such that $T(\vec{x}) = A\vec{x}$." Some of you have asked whether or not there are other matrices that represent the transformation. The answer is... sort of, yeah.

Notice that when we write $T(\vec{x}) = A\vec{x}$, we assume $\vec{x}$ is a vector in the standard basis. If we used a different basis... we would represent the linear transformation differently, i.e., with a different matrix.

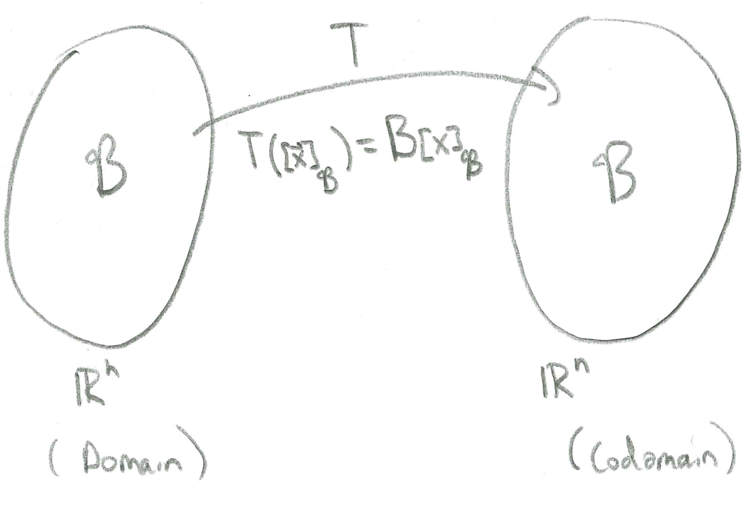
Think about it like this. Let S be the standard basis.

Then:



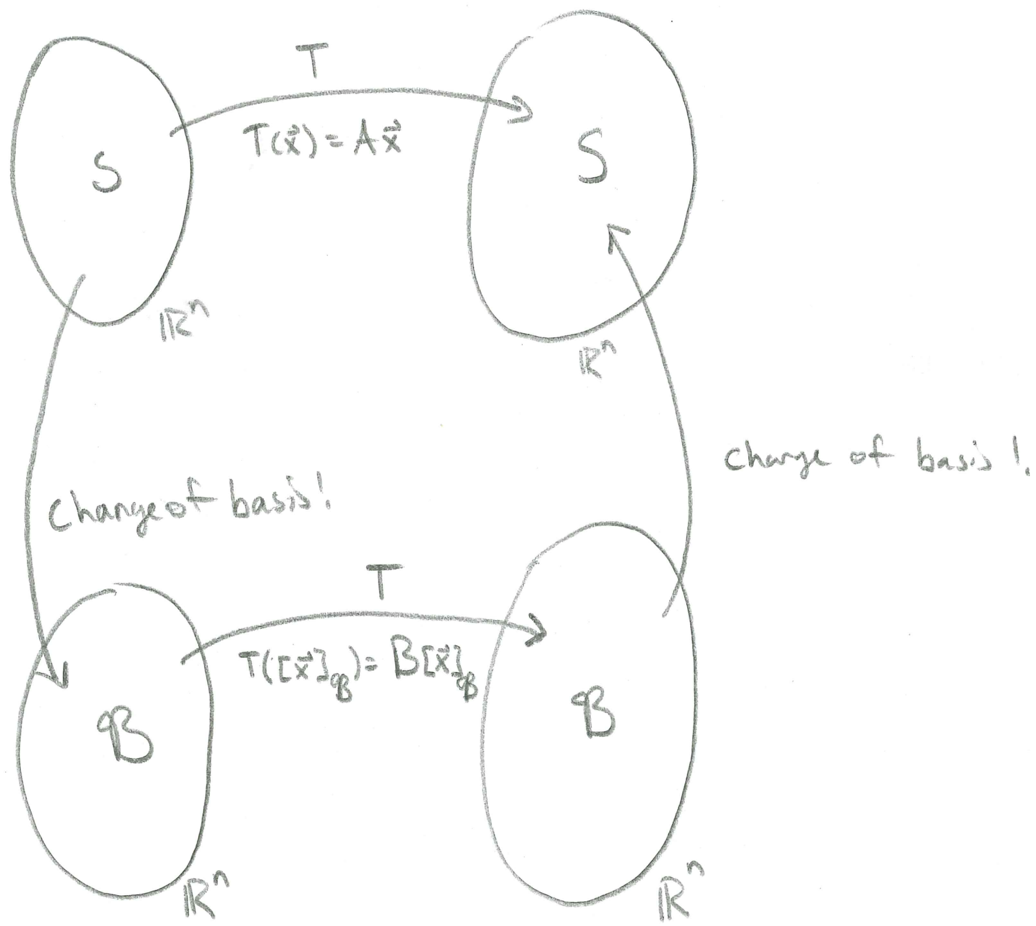
But, we might be interested in "representing" T using a different basis. Let $\mathcal{B}$ be some non-standard basis..

then:



If we choose a different bases, we would get a different matrix that represents the transformation.

Now, the obvious question becomes how are these two matrices related? If A and B both represent the same linear transformation, then we should have the following:



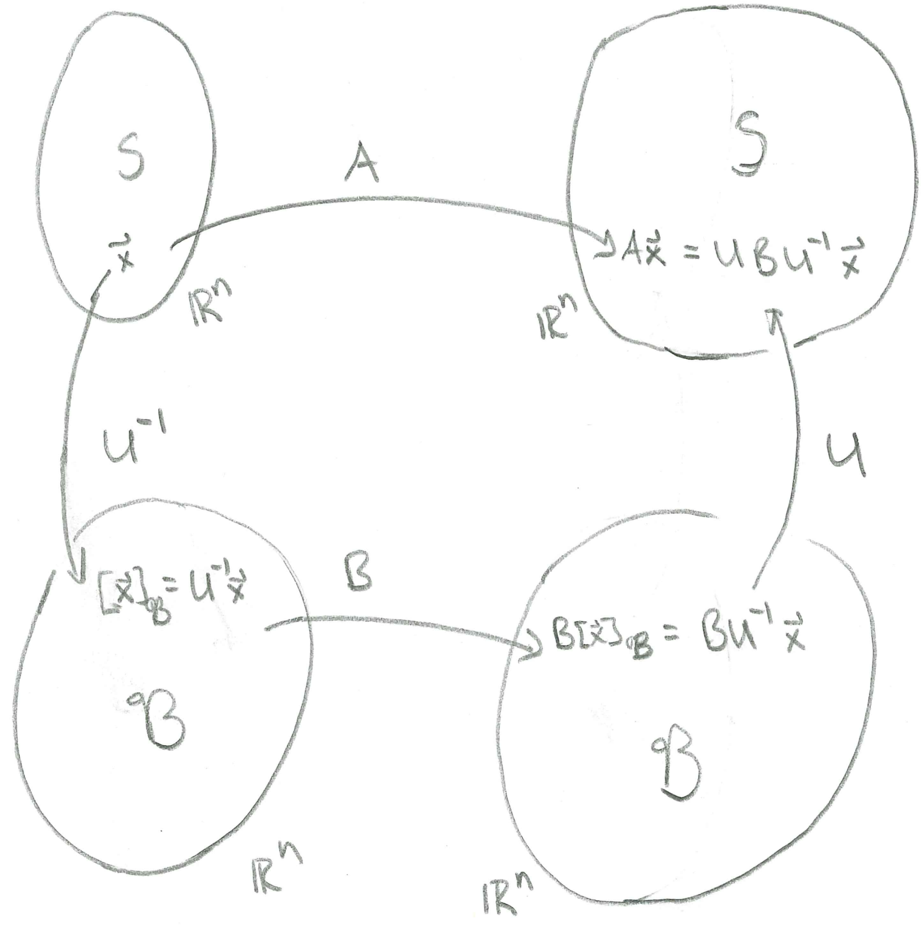
If we take a vector $\vec{x}$, then apply A, we get $A\vec{x}$. This needs to be the <u>same</u> thing as taking the vector $\vec{x}$, changing the basis to $\mathcal{B}$ to get $[\vec{x}]_{\mathcal{B}}$, then applying B (the same linear transformation!) to get $B[\vec{x}]_{\mathcal{B}}$, then changing the basis back to the standard basis.

Now, let $\mathcal{B} = \{\vec{u}_1, \ldots, \vec{u}_n\}$ be the non-standard basis, and let $U = [\vec{u}_1 \cdots \vec{u}_n]_{n \times n}$ be a matrix. Recall how we change bases (see 4.4):

$$① \quad \vec{x} = U[x]_{\mathcal{B}},$$

$$② \quad [\vec{x}]_{\mathcal{B}} = U^{-1}\vec{x}.$$

Now, we can update our picture:



And, if $A\vec{x} = UBU^{-1}\vec{x}$, then $\boxed{A = UBU^{-1}}$.

$$\underbrace{A}_{\substack{\text{matrix}\\ \text{with respect}\\ \text{to standard}\\ \text{basis}}} = U \underbrace{B}_{\substack{\text{matrix with}\\ \text{respect to}\\ \text{basis } \mathcal{B}.}} U^{-1}$$

We can read this formula as "applying A to a vector $\vec{x}$ in standard basis is the same as changing the basis to $\mathcal{B}$ (applying U^{-1}), then applying B, then changing the basis back." Let's see a simple example.

EXAMPLE 1

Show that $A = \begin{bmatrix} 3 & 4 \\ 5 & 6 \end{bmatrix}$ represents the same linear transformation in the standard basis as $B = \begin{bmatrix} 9 & 1 \\ 2 & 0 \end{bmatrix}$ does in the basis $\mathcal{B} = \left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \end{bmatrix} \right\}$.

To show this, let $U = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$, then $U^{-1} = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} \end{bmatrix}$

and we only need to show

$$A = U B U^{-1}$$

$$UBU^{-1} = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 9 & 1 \\ 2 & 0 \end{bmatrix}\begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} \end{bmatrix}$$

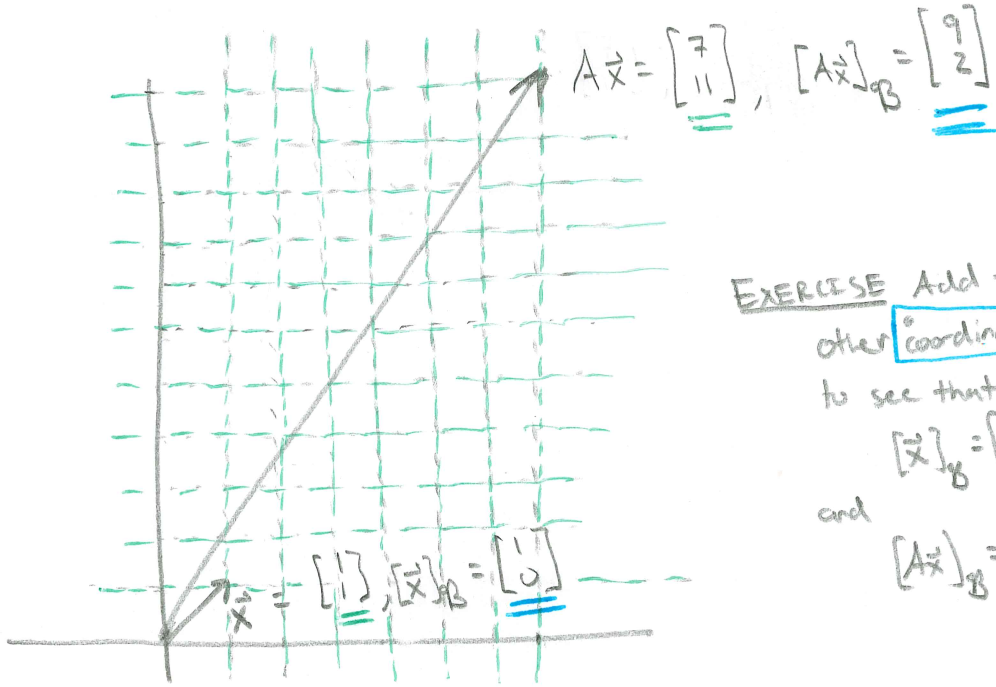
$$= \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 4 & 5 \\ 1 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 3 & 4 \\ 5 & 6 \end{bmatrix} = A.$$

☒

Perhaps that was not very enlightening. Let's try to make it a touch more concrete. Using the same matrices, let $\vec{x} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$. Then

$$A\vec{x} = \begin{bmatrix} 3 & 4 \\ 5 & 6 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \underline{\underline{\begin{bmatrix} 7 \\ 11 \end{bmatrix}}}.$$



EXERCISE Add the other "coordinate system" to see that

$$[\vec{x}]_{\mathcal{B}} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

and

$$[A\vec{x}]_{\mathcal{B}} = \begin{bmatrix} 9 \\ 2 \end{bmatrix}.$$

Notice that $\vec{x} = \underline{\underline{\begin{bmatrix} 1 \\ 1 \end{bmatrix}}}$ in the basis $\mathcal{B}$ is just $[\vec{x}]_{\mathcal{B}} = \underline{\underline{\begin{bmatrix} 1 \\ 0 \end{bmatrix}}}$.

If we apply B to $[\vec{x}]_{\mathcal{B}}$ we get:

$$B[\vec{x}]_{\mathcal{B}} = \begin{bmatrix} 9 & 1 \\ 2 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 9 \\ 2 \end{bmatrix}.$$

Notice that $[A\vec{x}]_{\mathcal{B}}$ is:

$$\begin{aligned} [A\vec{x}]_{\mathcal{B}} &= U^{-1}(A\vec{x}) \\ &= \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} \end{bmatrix} \begin{bmatrix} 7 \\ 11 \end{bmatrix} \\ &= \begin{bmatrix} 9 \\ 2 \end{bmatrix}. \end{aligned}$$

Okay, so hopefully that makes it a little more concrete. Here's the big question ... what does this have to do with diagonal matrices? And the answer is simple.. or rather, a better question: Given a matrix A, can we find a matrix representing the same linear transformation ... that is diagonal?

And the answer to that is... yes! Well, sometimes. We can provided the matrix has a set of eigenvectors which are a basis for $\mathbb{R}^n$. How does that work?

Check this out. Let A be an $n \times n$ matrix with n linearly independent eigenvectors, $\vec{v}_1, \ldots, \vec{v}_n$. Then span $\{\vec{v}_1, \ldots, \vec{v}_n\} = \mathbb{R}^n$ and we see that $\{\vec{v}_1, \ldots, \vec{v}_n\}$ make a basis. What is the matrix representing the same linear transformation as A, but in the basis $\{\vec{v}_1, \ldots, \vec{v}_n\}$? Remember, to each eigenvector there is a corresponding eigenvalue. Call them $\lambda_1, \ldots, \lambda_n$ and we have

$$\begin{aligned} A\vec{v}_1 &= \lambda_1 \vec{v}_1 \\ A\vec{v}_2 &= \lambda_2 \vec{v}_2 \\ &\vdots \\ A\vec{v}_n &= \lambda_n \vec{v}_n . \end{aligned}$$

We can compress this (the same way we compressed the set of matrices when figuring out how to find an inverse, see lecture notes for 3.3, p.):

$$\begin{aligned} A[\vec{v}_1 \cdots \vec{v}_n] &= [A\vec{v}_1 \;\; A\vec{v}_2 \;\; A\vec{v}_3 \cdots A\vec{v}_n] \\ &= [\lambda_1\vec{v}_1 \;\; \lambda_2\vec{v}_2 \;\; \lambda_3\vec{v}_3 \cdots \lambda_n\vec{v}_n] \\ &= [\vec{v}_1 \;\; \vec{v}_2 \cdots \vec{v}_n] \begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & & \vdots \\ \vdots & & \ddots & \\ 0 & \cdots & & \lambda_n \end{bmatrix} \end{aligned}$$

Let $D = \begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & & \vdots \\ \vdots & & \ddots & \\ 0 & \cdots & & \lambda_n \end{bmatrix}_{n\times n}$ be the diagonal matrix, and

let $U = [\vec{v}_1 \cdots \vec{v}_n]$, then

$$AU = UD$$

$$(*) \qquad \boxed{A = UDU^{-1}} \qquad (*)$$

Since U is a change of basis matrix, (see unifying theorem!) the matrix D is representing the same linear transformation as the matrix A, just in a very convenient basis! A basis of eigenvectors!

DEF An $n \times n$ matrix A is **diagonalizable** if there exist $n \times n$ matrices D and U, with D diagonal and U invertible, such that

$$\boxed{A = UDU^{-1}}.$$

REMARK The book calls the matrix U above "P", so if you see a formula of the form $A = PDP^{-1}$, it is just the same thing. $(*)$

That definition is a little silly since we have seen how to diagonalize a matrix:

<u>Diagonalizing an $n \times n$ matrix A</u>

- Pick n linearly independent eigenvectors $\vec{v}_1, \ldots, \vec{v}_n$ with corresponding eigenvalues $\lambda_1, \ldots, \lambda_n$.

- Set $U = [\vec{v}_1 \cdots \vec{v}_n]$, then

$$A = UDU^{-1}$$

where

$$D = \begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & & \\ \vdots & & \ddots & \\ 0 & & & \lambda_n \end{bmatrix}$$

is a diagonal matrix with eigenvalues along the diagonal.

<u>NOTE</u> It is <u>very</u> <u>important</u> that you pick λ_1 to be the eigenvalue corresponding to $\vec{v}_1$, the first column in the matrix U, and λ_j be the eigenvalue corresponding to $\vec{v}_j$ in the j^{th} column, etc. Otherwise, none of this will work!

Now, it is also true that if you have a matrix that is diagonalizable by the definition we gave (notice that the definition does not appeal to eigenvectors or eigenvalues) that the matrix U will always have columns that are a set of eigenvectors! For a proof of this, see p. 263 of the text.

(*) **THM** An $n \times n$ matrix A is diagonalizable if and only if A has eigenvectors that form a basis for $\mathbb{R}^n$.

This is interesting because it tells us the following:

(*) If there are not n linearly independent eigenvectors for a given matrix A, then A is <u>not</u> diagonalizable! (*)

In other words, not all matrices are diagonalizable.

Let's do a couple examples.

EXAMPLE 2

Find matrices U and D to diagonalize

$$A = \begin{bmatrix} 3 & 1 \\ -2 & 0 \end{bmatrix}$$

If we can find two linearly independent eigenvectors and their corresponding eigenvalues, we are golden. Let's start by finding eigenvalues:

$$0 = \det(A - \lambda I_2) = \begin{vmatrix} 3-\lambda & 1 \\ -2 & -\lambda \end{vmatrix}$$

$$= -\lambda(3-\lambda) + 2$$

$$= \lambda^2 - 3\lambda + 2$$

$$= (\lambda - 2)(\lambda - 1)$$

So, $\lambda_1 = 1$, $\lambda_2 = 2$ (each of multiplicity 1).

Now, lets find an eigenvector $\vec{v}_1$ for λ_1.

$$A - \lambda_1 I_2 = \begin{bmatrix} 3 & 1 \\ -2 & 0 \end{bmatrix} - 1 \cdot \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 2 & 1 \\ -2 & -1 \end{bmatrix}$$

Solving $(A - \lambda_1 I_2)\vec{v} = \vec{0}$ gives us:

$$\left[\begin{array}{cc|c} 2 & 1 & 0 \\ -2 & -1 & 0 \end{array}\right] \sim \left[\begin{array}{cc|c} 2 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

Let $\vec{v} = \begin{bmatrix} x \\ y \end{bmatrix}$, then $y = s$ (free variable) and $x = \frac{-1}{2}s$.

$$\vec{v} = \begin{bmatrix} -\frac{1}{2}s \\ s \end{bmatrix} = s \begin{bmatrix} -\frac{1}{2} \\ 1 \end{bmatrix}$$

Letting $s = 2$, (or anything but 0)! we get an eigenvector for λ_1:

$$\vec{v}_1 = 2\begin{bmatrix} -\frac{1}{2} \\ 1 \end{bmatrix} = \begin{bmatrix} -1 \\ 2 \end{bmatrix}$$

Next, for $\lambda_2 = 2$:

$$A - \lambda_2 I_2 = \begin{bmatrix} 3 & 1 \\ -2 & 0 \end{bmatrix} - 2\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 1 \\ -2 & -2 \end{bmatrix}.$$

Solving $(A - \lambda_2 I_2)\vec{v} = \vec{0}$ (letting $\vec{v} = \begin{bmatrix} x \\ y \end{bmatrix}$) gives us:

$$\left[\begin{array}{cc|c} 1 & 1 & 0 \\ -2 & -2 & 0 \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

So, $y = s$ (free variable) and $x = -s$. Thus

$$\vec{v} = \begin{bmatrix} -s \\ s \end{bmatrix} = s\begin{bmatrix} -1 \\ 1 \end{bmatrix},$$

and letting $s = 1$, we pick $\vec{v}_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$.

Now, define $U = [\vec{v}_1\ \vec{v}_2] = \begin{bmatrix} -1 & -1 \\ 2 & 1 \end{bmatrix}$, and $D = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$.

$\boxtimes$

Notice, $U^{-1} = \begin{bmatrix} 1 & 1 \\ -2 & -1 \end{bmatrix}$, and

we can check our answer!

$$UDU^{-1} = \begin{bmatrix} -1 & -1 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ -2 & -1 \end{bmatrix}$$

$$= \begin{bmatrix} -1 & -1 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ -4 & -2 \end{bmatrix}$$

$$= \begin{bmatrix} 3 & 1 \\ -2 & 0 \end{bmatrix} = A.$$

$\Rightarrow A = UDU^{-1}$ as needed. (nice!)

Let's do another example.

EXAMPLE 3

Find a 2x2 matrix A that has eigen values $\lambda_1 = -1$, $\lambda_2 = 2$ and corresponding eigenvectors $\vec{u}_1 = \begin{bmatrix} 5 \\ 3 \end{bmatrix}$ and $\vec{u}_2 = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$.

To do this ... well, we already know the diagonalized matrix!

$$D = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & 2 \end{bmatrix}.$$

We also know what U needs to be:

$$U = [\vec{u}_1 \ \ \vec{u}_2] = \begin{bmatrix} 5 & 3 \\ 3 & 2 \end{bmatrix},$$

So to find A, we need only to solve

$$\boxed{A = UDU^{-1}}.$$

Let's find U^{-1}. $U^{-1} = \frac{1}{10-9}\begin{bmatrix} 2 & -3 \\ -3 & 5 \end{bmatrix} = \begin{bmatrix} 2 & -3 \\ -3 & 5 \end{bmatrix}.$

So,

$$A = UDU^{-1} = \begin{bmatrix} 5 & 3 \\ 3 & 2 \end{bmatrix}\begin{bmatrix} -1 & 0 \\ 0 & 2 \end{bmatrix}\begin{bmatrix} 2 & -3 \\ -3 & 5 \end{bmatrix}$$

$$= \begin{bmatrix} 5 & 3 \\ 3 & 2 \end{bmatrix}\begin{bmatrix} -2 & 3 \\ -6 & 10 \end{bmatrix}$$

$$= \begin{bmatrix} -28 & 45 \\ -18 & 29 \end{bmatrix}.$$

$\boxtimes$

<u>EXERCISE</u> Find eigenvalues for $A = \begin{bmatrix} -28 & 45 \\ -18 & 29 \end{bmatrix}$. Show that $\vec{u}_1$ and $\vec{u}_2$ above are eigenvectors corresponding to these eigenvalues.

Here are some fun facts about matrices ... which are important when it comes to determining if a matrix can be diagonalized.

THM If $\{\lambda_1, \ldots, \lambda_k\}$ are distinct eigenvalues of a matrix A, then any set of associated eigenvectors $\{\vec{u}_1, \ldots, \vec{u}_k\}$ is linearly independent.

For a proof, see p. 264 in the text.

EXERCISE After looking at the proof, go back to the theorem. Try thinking about it geometrically ... can you see why it must be true from a geometric standpoint? [HINT: Try looking at $A = \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}$, the matrix in multiple examples in 6.1. If λ_1 and λ_2 had eigenvectors that were linearly dependent ... what would that mean?]

EXERCISE Diagonalize $A = \begin{bmatrix} 1 & 1 & 1 \\ -2 & -2 & -1 \\ 0 & 0 & -1 \end{bmatrix}$.

Notice that A is not invertible, but you can diagonalize it! And when you do, the diagonal matrix D is not invertible.

[HINT: See Example 4 on page 265 if you get stuck.]

Next, let's give a condition for precisely when a matrix with real eigenvalues is diagonalizable:

THM Suppose that an $n \times n$ matrix A has only real eigenvalues. Then A is diagonalizable if and only if the dimension of each eigenspace is equal to the multiplicity of the corresponding eigenvalue.

This follows from facts we already know are true. First, A is diagonalizable if and only if A has n linearly independent eigenvectors. Each eigenspace has dimension less than or equal to the multiplicity of the corresponding eigenvalue. The sum of the multiplicities of the eigenvalues is n (characteristic polynomial is degree n). Since vectors from distinct eigenspaces are linearly independent (by the theorem on the previous page), the dimension of each eigenspace must be the same as the multiplicity, otherwise there won't be $\underline{\underline{n}}$ linearly independent eigenvectors.

For example, if the characteristic equation for a 3×3 matrix was $-(\lambda+2)^2(\lambda-2)$, and $\lambda_1 = -2$ has only a 1-dimensional eigenspace, we can only get two linearly independent eigenvectors! Ah! So the matrix would not be diagonalizable!

EXAMPLE 4

Can we diagonalize $A = \begin{bmatrix} 3 & 6 & 5 \\ 3 & 2 & 3 \\ -5 & -6 & -7 \end{bmatrix}$?

Find eigenvalues first:

$$0 = \det(A - \lambda I_3) = \begin{vmatrix} 3-\lambda & 6 & 5 \\ 3 & 2-\lambda & 3 \\ -5 & -6 & -7-\lambda \end{vmatrix}$$

$$= -(\lambda+2)^2(\lambda-2).$$

Let's try to find two linearly independent eigenvectors for $\lambda_1 = -2$ (knowing that $\lambda_2 = 2$ has an eigenspace which is one-dimensional .. it can't be greater than the multiplicity, and it can't be less than 1 otherwise there are no corresponding eigenvectors).

For $\lambda_1 = -2$, let $\vec{v} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ and solve

$$\underbrace{(A - (-2)I_2)}\vec{v} = 0$$

$$\left[\begin{array}{ccc|c} 5 & 6 & 5 & 0 \\ 3 & 4 & 3 & 0 \\ -5 & -6 & -5 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} 5 & 6 & 5 & 0 \\ 0 & \frac{2}{5} & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

So, we see $z = s$ (free variable), $y = 0$, and $x = -s$. Thus

$$\vec{v} = \begin{bmatrix} -s \\ 0 \\ s \end{bmatrix} = s \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \ldots$$

and we see the eigenspace is one-dimensional! So we can only get one linearly independent eigenvector here. Since $\lambda_2 = 2$ is a distinct eigenvalue, we will get one more eigenvector, but this will give us at most 2 eigenvectors! And we would need 3 to span $\mathbb{R}^3$.

Thus A is <u>not</u> invertible. $\boxtimes$

The following fact is an immediate consequence of the last two theorems:

(*) <u>THM</u> If A is an $n \times n$ matrix with n distinct real eigenvalues, then A is diagonalizable. (*)

Let's see one last application of diagonalizing a matrix. Let's consider powers of a diagonalizable matrix:

Let A be diagonalizable. Then $A = UDU^{-1}$ for some diagonal matrix D. Then

(*)

$$A^k = (UDU^{-1})^k$$

$$= \underbrace{UD\overbrace{U^{-1}\cdot U}^{I_n}D\overbrace{U^{-1}\cdots U}^{I_n}DU^{-1}}_{k \text{ times}}$$

$$= U\underbrace{D\cdot D\cdot D\cdots D}_{k \text{ times}}U^{-1}$$

$$= UD^kU^{-1}.$$

EXAMPLE 5

Let $A = \begin{bmatrix} \frac{1}{3} & \frac{2}{9} \\ \frac{2}{3} & \frac{7}{9} \end{bmatrix}$. Find U and D so that $A = UDU^{-1}$. Use this to give a formula for A^k.

First, confirm $\lambda_1 = 1$ and $\lambda_2 = \frac{1}{9}$ are eigenvalues with corresponding eigenvectors $u_1 = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$ and $u_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$.

Then, $U = \begin{bmatrix} 1 & -1 \\ 3 & 1 \end{bmatrix}$, $D = \begin{bmatrix} 1 & 0 \\ 0 & \frac{1}{9} \end{bmatrix}$. Then $A = UDU^{-1}$.

To compute A^k, notice:

$$A^k = U D^k U^{-1}$$

diagonal! $\qquad U^{-1}$

$$= \begin{bmatrix} 1 & -1 \\ 3 & 1 \end{bmatrix} \begin{bmatrix} 1^k & 0 \\ 0 & (\frac{1}{9})^k \end{bmatrix} \begin{bmatrix} \frac{1}{4} & \frac{1}{4} \\ -\frac{3}{4} & \frac{1}{4} \end{bmatrix}$$

$$= \frac{1}{4} \begin{bmatrix} 1 + 3(\frac{1}{9})^k & 1 - (\frac{1}{9})^k \\ 3 - 3(\frac{1}{9})^k & 3 + (\frac{1}{9})^k \end{bmatrix}.$$

☺

<u>EXERCISE</u> Go back to p.3 of the 6.1 lecture notes.

Where should $\vec{v}$ be mapped?

[<u>Hint</u>: For $\vec{v}_2$, notice, we can write $\vec{v}_2$ as

$\vec{v}_{2(a)} + \vec{v}_{2(b)} = \vec{v}_2$, where $\vec{v}_{2(a)}$ points

along E_1 and $\vec{v}_{2(b)}$ points along E_2.

Then $A\vec{v}_2 = A(\vec{v}_{2(a)} + \vec{v}_{2(b)})$

$= A\vec{v}_{2(a)} + A\vec{v}_{2(b)}$ $\quad$ $\vec{v}_{2(a)}$ is in E_1, i.e. it is an eigenvector!

$= 7\,\vec{v}_{2(a)} + {}^{-}2\,\vec{v}_{2(b)}$.

Can you estimate $\vec{v}$ in the same way?]

<u>EXERCISE</u> Show that for a diagonalizable matrix A, the determinant of A is equal to the product of its eigenvalues (up to multiplicity).

[<u>HINT</u>: $A = UDU^{-1}$, $D = \begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \ddots & & \\ \vdots & & & \\ 0 & & & \lambda_n \end{bmatrix}$. $\det(D) = ?$]

EXERCISE Review the Chapter 6 conceptual problems.

You can find these on the course website under "Other Materials".

In addition, watch the Chapter 6 video.