In the next few lectures, we will be developing a very useful function, one whose input is a matrix and output a real number. As we develop the formula for the function, it may seem haphazard, or only able to serve one purpose (identifying if a matrix is invertible), but it turns out to have many, many useful properties

5.1/2 The Determinant Function and its Properties

In some sense, we can think of matrices as generalizations of real numbers; real numbers follow all of the rules of 1×1 matrices. So if we want to find a function that will determine if a matrix is invertible, we could start with asking the question when is a real number, a 1×1 matrix, invertible? In other words, when r is a real number, when can we find some "r^{-1}" such that $r \cdot r^{-1} = 1$? Here, our notation for inverse, $\square^{-1}$, is quite suggestive (in fact it is why we use it). We need $r^{-1} = \frac{1}{r}$ so that

$$r \cdot r^{-1} = r \cdot \frac{1}{r} = 1.$$

So when is r invertible? When $r \neq 0$.

Now, if we want a function on a 1×1 matrix, we really only have one choice: Let

$$[r]_{1\times1},$$

call the function "det" for determinant, and

$$\det([r]_{1\times1}) = r.$$

DEF Let $A = [a_{11}]$ be a 1×1 matrix. Then the determinant of A is given by

$$\boxed{\det(A) = a_{11}}.$$

THM The 1×1 matrix $A = [a_{11}]$ is invertible if and only if $a_{11} \neq 0$, i.e. $\det(A) \neq 0$.

So, in the case of real numbers, or 1×1 matrices, as long as the determinant is non-zero, the matrix is invertible. Our next goal is to try to generalize this to 2×2 matrices. Do we know of a condition that when non-zero, means a 2×2 matrix is invertible?

REM We have a formula for the inverse of a 2×2 matrix.

If $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, then $A^{-1} = \dfrac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$

$\Rightarrow$ A will be invertible if $ad-bc \neq 0$. (!)

DEF Let A be the 2×2 matrix

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}.$$

The **determinant** of A is given by

$$\boxed{\det(A) = a_{11}a_{22} - a_{12}a_{21}}.$$

THM A 2×2 matrix A is invertible if and only if $\det(A) \neq 0$.

VISUAL METHOD (for 2×2)

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \rightsquigarrow \det(A) = a_{11}a_{22} - a_{12}a_{21}.$$

NOTATION Let $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$. We sometimes denote the determinant as follows:

$$\boxed{\det(A) = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}} \quad (*)$$

lines instead of brackets!

EXAMPLE 1 Let $A = \begin{bmatrix} 3 & 5 \\ -1 & 4 \end{bmatrix}$. Is A invertible?

We can just check to see if $\det(A) \neq 0$:

$$\det(A) = \begin{vmatrix} 3 & 5 \\ -1 & 4 \end{vmatrix} = 12 - {}^{-}5 = 17 \neq 0$$

Thus, A is invertible.

Now comes the tricky part... how would we keep generalizing to 3x3, and eventually $n \times n$? First, we have to remember where our 2x2 inversion formula came from: one of the exercises in 3.3 was deriving the inverse for a general 2x2 matrix using the method we developed for finding an inverse. We'll do that exercise here. First, let

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix},$$

then compute the inverse. (Assume $a \neq 0$, otherwise swap rows first!)

$$\left[\begin{array}{cc|cc} a & b & 1 & 0 \\ c & d & 0 & 1 \end{array}\right] \underset{-\frac{c}{a}R_1 + R_2 \to R_2}{\sim} \left[\begin{array}{cc|cc} a & b & 1 & 0 \\ 0 & d-\frac{bc}{a} & -\frac{c}{a} & 1 \end{array}\right]$$

$$\underset{aR_2 \to R_2}{\sim} \left[\begin{array}{cc|cc} a & b & 1 & 0 \\ 0 & (ad-bc) & -c & a \end{array}\right]$$

$$\underset{\frac{-b}{ad-bc} \cdot R_2 + R_1 \to R_1}{\sim} \left[\begin{array}{cc|cc} a & 0 & 1+\frac{bc}{ad-bc} & \frac{-ab}{ad-bc} \\ 0 & (ad-bc) & -c & a \end{array}\right]$$

$$= \left[\begin{array}{cc|cc} a & 0 & \frac{ad-\cancel{bc}+\cancel{bc}}{(ad-bc)} & \frac{-ab}{ad-bc} \\ 0 & (ad-bc) & -c & a \end{array}\right]$$

$$\underset{\substack{\frac{1}{a}R_1 \to R_1 \\ \frac{1}{(ad-bc)}R_2 \to R_2}}{\sim} \left[\begin{array}{cc|cc} 1 & 0 & \frac{d}{(ad-bc)} & \frac{-b}{(ad-bc)} \\ 0 & 1 & \frac{-c}{(ad-bc)} & \frac{a}{(ad-bc)} \end{array}\right]$$

Thus, $A^{-1} = \begin{bmatrix} \frac{d}{ad-bc} & \frac{-b}{ad-bc} \\ \frac{-c}{ad-bc} & \frac{a}{ad-bc} \end{bmatrix} = \frac{1}{(ad-bc)} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}.$

Now, the idea is to use this same procedure on a 3x3 matrix, then identify the condition that makes it invertible. The book has an elegant way of identifying this condition (see p. 210 if you are interested). We will just identify that condition:

DEF Let A be a 3×3 matrix,

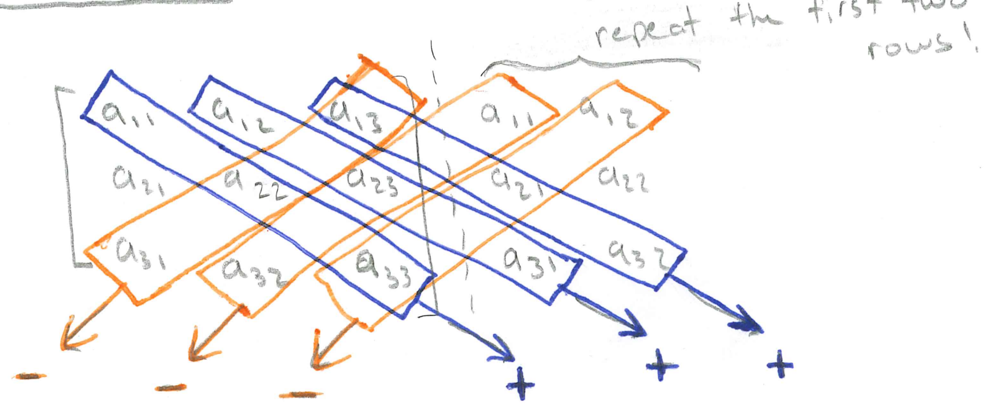
$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}.$$

The $\boxed{\text{determinant}}$ of A is given by

$$\boxed{\begin{aligned} \det(A) = {} & a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} \\ & - a_{11}a_{23}a_{32} - a_{12}a_{21}a_{33} - a_{13}a_{22}a_{31} \end{aligned}} \quad (*)$$

So... ugh.. right? Actually, it is not that bad. Here is a visual shortcut to the formula:

VISUAL METHOD (3x3 matrices)



When you are working problems, using this to compute a determinant is often the easiest way to do it.

WARNING The visual method DOES NOT work for 4x4 or higher dimensional matrices. This is a special method specifically for 3x3 matrices.

EXAMPLE 2 Let $A = \begin{bmatrix} -3 & 1 & 2 \\ 5 & 5 & -8 \\ 4 & 2 & -5 \end{bmatrix}$. Is A invertible?

Again, we need only compute $\det(A)$. We will use the visual method:

$$\begin{bmatrix} -3 & 1 & 2 \\ 5 & 5 & -8 \\ 4 & 2 & -5 \end{bmatrix} \begin{matrix} -3 & 1 \\ 5 & 5 \\ 4 & 2 \end{matrix} \Rightarrow \det(A) = (-3)(5)(-5) + (1)(-8)(4) + (2)(5)(2)$$

$$- (2)(5)(4) - (-3)(-8)(2) - (1)(5)(-5)$$

$$= (75 - 32 + 20) - (40 + 48 - 25)$$

$$= 0.$$

$- \quad - \quad - \quad + \quad + \quad +$

Thus A is not invertible.

Next, we continue our quest to generalize to $n \times n$ matrices. We could come up with a determinant for each dimension one at a time, but there is actually a way of looking at the determinant of a 3×3 matrix that will more naturally generalize. We need a definition to do this.

DEF Let A be an $n \times n$ matrix, and let $\boxed{M_{ij}}$ denote the $(n-1) \times (n-1)$ matrix that we get from A after deleting the row and column containing a_{ij}. The determinant of M_{ij}, $\det(M_{ij})$, is called the $\boxed{\text{minor}}$ of a_{ij}.

we only know what this is for 2×2, 3×3 (and 1×1) matrices right now!

DEF The $\boxed{\text{cofactor}}$ of a_{ij} is given by

$$C_{ij} = (-1)^{i+j} \det(M_{ij}).$$

the value of $(-1)^{i+j}$ is ± 1 ... whether it is $+$ or $-$ in the matrix is indicated below

(*) PICTURE

$(-1)^{1+1}$

$(-1)^{2+1}$

$$\begin{bmatrix} + & - & + & - & \cdots \\ - & + & - & + & \cdots \\ + & - & + & - & \cdots \\ - & + & - & + & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}$$

Now, let's think about $\det(A)$, $A_{3\times3}$, using our new terminology:

Let $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$.

$$\det(A) = a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} - a_{11}a_{23}a_{32} - a_{12}a_{21}a_{33} - a_{13}a_{22}a_{31}$$

Notice that a_{11} shows up in two terms, a_{12} shows up in two terms, and a_{13} shows up in two terms. These represent a row, so lets see what happens if we factor...

$$= a_{11}(a_{22}a_{33} - a_{23}a_{32}) + a_{12}\underbrace{(a_{23}a_{31} - a_{21}a_{33})} + a_{13}(a_{21}a_{32} - a_{22}a_{31})$$

$$= a_{11}\underbrace{(a_{22}a_{33} - a_{23}a_{32})}_{\det(M_{11})} - a_{12}(a_{21}a_{33} - a_{23}a_{31}) + a_{13}\underbrace{(a_{21}a_{32} - a_{22}a_{31})}_{\det(M_{13})}$$

$$\underbrace{\begin{bmatrix} \cancel{a_{11}} & \cancel{a_{12}} & \cancel{a_{13}} \\ \cancel{a_{21}} & a_{22} & a_{23} \\ \cancel{a_{31}} & a_{32} & a_{33} \end{bmatrix}}_{M_{11}} \qquad \underbrace{\begin{bmatrix} \cancel{a_{11}} & \cancel{a_{12}} & \cancel{a_{13}} \\ a_{21} & \cancel{a_{22}} & a_{23} \\ a_{31} & \cancel{a_{23}} & a_{33} \end{bmatrix}}_{M_{12}}$$

EXERCISE Find M_{13} for A as above.

Now, notice that for a 3×3 matrix, the sign in the cofactor is given by:

$$\begin{bmatrix} + & - & + \\ - & + & - \\ + & - & + \end{bmatrix}_{3\times3}$$

So,
$$\det(A) = a_{11}\cdot\det(M_{11}) + a_{12}\cdot(-1)\det(M_{12}) + a_{13}\det(M_{13})$$
$$= a_{11}C_{11} + a_{12}C_{12} + a_{13}C_{13}, \quad (*)$$

where C_{ij} denotes the cofactor defined above.

(*) THM For A a 3×3 matrix,

$$\det(A) = a_{11}C_{11} + a_{12}C_{12} + a_{13}C_{13},$$

where C_{11}, C_{12}, C_{13} are the cofactors of a_{11}, a_{12}, a_{13}, respectively.

It is exactly this construction that generalizes for us! (There are many other ways to approach this. If you learn any group theory, there is a very elegant formulation of the determinant using some of that machinery.)

DEF Let A be an $n \times n$ matrix,

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & & a_{nn} \end{bmatrix}.$$

The $\boxed{\text{determinant}}$ of A is

$$\boxed{\det(A) = a_{11}C_{11} + a_{12}C_{12} + \cdots + a_{1n}C_{1n}}$$

where $C_{11}, \ldots, C_{1n}$ are cofactors of $a_{11}, \ldots, a_{1n}$, respectively. When $n=1$, $A = [a_{11}]$ and $\det(A) = a_{11}$.

EXERCISE Use this theorem to generate a formula for a 2×2 matrix. Can you reverse engineer the formula for a 3×3 matrix?

EXAMPLE 3 Let $A = \begin{bmatrix} -3 & 1 & 2 \\ 5 & 5 & -8 \\ 4 & 2 & -5 \end{bmatrix}$. Is A invertible?

This is the same as example 2, but this time, we will use the new definition above.

$$\det(A) = a_{11}C_{11} + a_{12}C_{12} + a_{13}C_{13}$$

$$C_{11} = (-1)^{1+1}\det(M_{11}), \quad M_{11} \Rightarrow \begin{bmatrix} -3 & 1 & 2 \\ 5 & 5 & -8 \\ 4 & 2 & -5 \end{bmatrix}$$

so,

$$M_{11} = \begin{bmatrix} 5 & -8 \\ 2 & -5 \end{bmatrix}$$

$$C_{11} = (+1)\cdot\begin{vmatrix} 5 & -8 \\ 2 & -5 \end{vmatrix} = -25-(-16) = -9$$

$$C_{12} = (-1)^{1+2}\det(M_{12}), \quad M_{12} \Rightarrow \begin{bmatrix} -3 & 1 & 2 \\ 5 & 5 & -8 \\ 4 & 2 & -5 \end{bmatrix}$$

so,

$$M_{12} = \begin{bmatrix} 5 & -8 \\ 4 & -5 \end{bmatrix}$$

$$C_{12} = (-1)\cdot\begin{vmatrix} 5 & -8 \\ 4 & -5 \end{vmatrix} = (-1)(-25-(-32)) = -7$$

Similarly, $C_{13} = (-1)^{1+3}\cdot\begin{vmatrix} 5 & 5 \\ 4 & 2 \end{vmatrix} = -10$

Then

$$\begin{aligned} \det(A) &= (-3)\cdot(-9) + (1)(-7) + 2(-10) \\ &= 27 - 7 - 20 \\ &= 0. \end{aligned}$$

So A is (still) not invertible!

Next, if you go back to page 9 where we decided to factor out entries from the first row .. the choice of the first row was arbitrary. I could have factored entries from the second row or first column .. or third column. It turns out that the way we defined "minor" and "cofactor" will enable us to show!

(*) **THM** Let A be the $n \times n$ matrix

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & & \vdots \\ \vdots & \vdots & \ddots & \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix}.$$

Then

(a) $\boxed{\det(A) = a_{i1} C_{i1} + a_{i2} C_{i2} + \cdots + a_{in} C_{in}}$

(expand across row i)

(b) $\boxed{\det(A) = a_{1j} C_{1j} + a_{2j} C_{2j} + \cdots + a_{nj} C_{nj}}$

(expand down column j)

where C_{ij} denotes the cofactor of a_{ij}.

REMARK The formulas that come from this theorem are called cofactor expansions.

PROPERTIES of the determinant function

EXAMPLE 4 $\det(I_2) = ?$

$$I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = 1 \implies \det(I_2) = 1$$

EXAMPLE 5 $\det(I_3) = ?$

$$\begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix} = 1 \implies \det(I_3) = 1$$

THM $\det(I_n) = 1$

EXERCISE Can you show this for an arbitrary $n \times n$ matrix? It is a little tricky. Try it first for I_4, then see if you can show it for I_n.

EXAMPLE 6 Let $A = \begin{bmatrix} 2 & 3 & -1 \\ 4 & 2 & 0 \\ 5 & -2 & -4 \end{bmatrix}$.

Compute $\det(A)$ and $\det(A^T)$.

$\det(A) = 50$.

$A^T = \begin{bmatrix} 2 & 4 & 5 \\ 3 & 2 & -2 \\ -1 & 0 & -4 \end{bmatrix}$, $\det(A^T) = 50$.

THM Let A be a square matrix. Then $\det(A) = \det(A^T)$.

How would you show this? Notice,

$$\det(A) = a_{11}C_{11} + a_{12}C_{12} + \cdots + a_{1n}C_{1n}$$

(expansion across first row.)

and

$$\det(A) = a_{11}C_{11} + a_{21}C_{21} + \cdots + a_{n1}C_{n1}$$

(expansion down first column)

But notice, $\det(A^T) = a_{11}C_{11} + a_{21}C_{21} + \cdots + a_{n1}C_{n1}$

also expansion across first row of A^T!

EXAMPLE 7

Let $A = \begin{bmatrix} 2 & -4 \\ -1 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 3 & -1 \\ 2 & 1 \end{bmatrix}$

Compute $\det(AB)$ and $\det(A)\cdot\det(B)$.

$\det(A) = 2 - 4 = -2$

$\det(B) = 3 - (-2) = 5$

so $\det(A)\cdot\det(B) = -2\cdot 5 = -10$.

$$AB = \begin{bmatrix} 2 & -4 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} 3 & -1 \\ 2 & 1 \end{bmatrix} = \begin{bmatrix} -2 & -6 \\ -1 & 2 \end{bmatrix}$$

$\det(AB) = -4 - 6 = -10$.

THM If A and B are both $n\times n$ matrices, then

$$\det(AB) = \det(A)\cdot\det(B).$$

REMARK This is one of the most important properties of the determinant function!

EXERCISE Suppose A is an invertible $n\times n$ matrix and B a singular matrix. Show that AB can never be invertible.

$(n\times n)$

Now, notice:

$$\det(AA^{-1}) = \det(A)\cdot\det(A^{-1}).$$

But we already know

$$\det(I_n) = 1$$

So,

$$1 = \det(I_n) = \det(AA^{-1}) = \det(A)\cdot\det(A^{-1})$$

thus,

THM For A an $n \times n$ invertible matrix,

$$\det(A^{-1}) = \frac{1}{\det(A)}.$$

EXERCISE Let $A = \begin{bmatrix} 2 & -1 & 4 \\ -6 & 3 & -3 \\ 1 & 5 & 0 \end{bmatrix}$. Confirm $\det(A^{-1}) = \frac{1}{\det(A)}$.

Now, let's ask a slightly different question. When we took a matrix and did Gauss-Jordan elimination, we modified the matrix by using elementary row operations. How do the elementary row operations influence the determinant?

EXAMPLE 8

Suppose $A = \begin{bmatrix} 2 & -1 & 4 \\ -6 & 3 & -3 \\ 1 & 5 & 0 \end{bmatrix}$. Compute det(A).

Then compute the determinants of each equivalent matrix listed below.

(a) B, where B comes from interchanging Row 1 and Row 3 of A.
$(R_1 \leftrightarrow R_3)$

(b) C, where C comes from multiplying Row 2 by $\frac{1}{3}$.
$(\frac{1}{3}R_2 \rightarrow R_2)$

(c) D, where D comes from adding -2 times row 3 to row 1.
$(-2R_3 + R_1 \rightarrow R_1)$

expanded along row 1.

First, $\det(A) = 2(0+15) + 1(0+3) + 4(-30-3)$

$= 30 + 3 - 132$

$= -99.$

Then

$$B = \begin{bmatrix} 1 & 5 & 0 \\ -6 & 3 & -3 \\ 2 & -1 & 4 \end{bmatrix}$$

and $\det(B) = 1(12-3) - 5(-24+6) + 0$

$= 9 + 5(18)$

$= 99.$

This example is not an accident. We can actually prove the following theorem.

THM Let A be a square matrix.

1. Suppose that B is produced by interchanging two rows of A. Then $\det(B) = -\det(A)$

2. Suppose that B is produced by multiplying a row of A by c. Then $\det(B) = c\det(A)$.

3. Suppose that B is produced by adding a multiple of one row of A to another. Then $\det(B) = \det(A)$.

We won't prove this here, but if you are interested, the proof of this theorem is given on p. 230 in the text (under the heading "Proof of Theorem 5.13").

There are a few nifty applications of the preceding theorem, especially when we combine it with the fact that we can expand cofactors down any row and any column. (This is the theorem on p. 13 of these lecture notes.)

So, interchanging two rows introduced a negative sign.

Next,

$$C = \begin{bmatrix} 2 & -1 & 4 \\ -2 & 1 & -1 \\ 1 & 5 & 0 \end{bmatrix}$$

and

$$\begin{aligned} \det(C) &= 2(0+5) + 1(0+1) + 4(-10-1) \\ &= 11 - 44 \\ &= -33. \end{aligned}$$

So, multiplying a row by $\frac{1}{3}$ multiplied the determinant by $\frac{1}{3}$!

Finally,

$$D = \begin{bmatrix} 0 & -11 & 4 \\ -6 & 3 & -3 \\ 1 & 5 & 0 \end{bmatrix}$$

and

$$\begin{aligned} \det(D) &= 0 + 11(0+3) + 4(-30-3) \\ &= 33 + {}^{-}4(33) \\ &= {}^{-}3(33) \\ &= {}^{-}99. \end{aligned}$$

So, adding a multiple of a row to another seems to have no effect on the determinant.

$\boxtimes$

EXAMPLE 9

Show that $\det(A) = 0$ and $\det(B) = 0$, where

$$A = \begin{bmatrix} 3 & 0 & 0 & 2 \\ 0 & 4 & 0 & 5 \\ 9 & 0 & 0 & 7 \\ 1 & 1 & 0 & 0 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} -1 & 4 & 2 & 0 \\ 6 & 2 & 5 & -4 \\ -1 & 4 & 2 & 0 \\ 0 & 8 & 4 & 7 \end{bmatrix}.$$

For A, we will appeal to the theorem on p. 13 by expanding down the 3rd column:

$$\det(A) = 0 \cdot C_{13} + 0 \cdot C_{23} + 0 \cdot C_{33} + 0 \cdot C_{34}$$
$$= 0.$$

For B, notice that row 1 and row 3 are identical. This means if we perform a row operation, we'll get a zero row:

$$B = \begin{bmatrix} -1 & 4 & 2 & 0 \\ 6 & 2 & 5 & -4 \\ -1 & 4 & 2 & 0 \\ 0 & 8 & 4 & 7 \end{bmatrix} \overset{(-R_1 + R_3 \to R_3)}{\sim} \begin{bmatrix} -1 & 4 & 2 & 0 \\ 6 & 2 & 5 & -4 \\ 0 & 0 & 0 & 0 \\ 0 & 8 & 4 & 7 \end{bmatrix} =: \tilde{B}$$

Then, by the Theorem on the previous page (part ③), we know $\det(B) = \det(\tilde{B})$.

Then, by th Theorem on p. 13, we can expand across the third row, and we get

$$\det(\tilde{B}) = 0 \cdot C_{31} + 0 \cdot C_{32} + 0 \cdot C_{33} + 0 \cdot C_{34}$$
$$= 0.$$

Thus, $\det(B) = 0$.

$\square$

In fact, you should notice the following:

THM Let A be a square matrix.

① If A has a row or column of zeros, then $\det(A) = 0$.

② If A has two identical rows or columns, then $\det(A) = 0$.

REMARK Notice that we should have expected ②. In fact, we should expect more! the determinant was constructed (in our notes) to identify when a matrix is invertible. If there is any linear dependency in the column vectors, this means the matrix is not invertible, which means $\det(A) = 0$. In fact, for square matrices, if there is any linear dependency in the row vectors, $\det(A) = 0$. Remember that $\dim(\mathrm{col}(A)) = \dim(\mathrm{row}(A))$! i.e. Dependency in the row vector implies dependency in the column vector!

EXERCISE Compute determinants of the following matrices.
Leverage the locations of 0 and look for dependencies.
Justify your answer!

1) $A = \begin{bmatrix} 2 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 4 & 1 & 0 & 2 \\ 0 & 1 & 2 & 3 \end{bmatrix}$

2) $A = \begin{bmatrix} 1 & 4 & 3 & 1 & 4 \\ 3 & 2 & 4 & 3 & 2 \\ 0 & 1 & 6 & 0 & -1 \\ 2 & -1 & 1 & 2 & 1 \\ 1 & 2 & 0 & 1 & -1 \end{bmatrix}$

3) $A = \begin{bmatrix} 2 & 1 & 6 & 2 \\ 3 & -2 & 4 & 1 \\ 2 & 1 & 6 & 2 \\ 3 & 5 & 2 & 4 \end{bmatrix}$

4) $A = \begin{bmatrix} 3 & 4 & 5 & 7 \\ 0 & -2 & 5 & -9 \\ 0 & 0 & 1 & 6 \\ 0 & 0 & 0 & 5 \end{bmatrix}$

[SOLUTION: 1) 0, 2) 0, 3) 0, 4) -30.]

EXERCISE

Find a general formula for the determinant of A when A is upper triangular. Find a general formula for the determinant of A if A is lower triangular.

[HINT: For A upper triangular, $\det(A)$ will be the product of the diagonal elements.]

EXERCISE

Recall the matrix $V = \begin{bmatrix} 1 & x_1 & x_1^2 & \cdots & x_1^{n-1} \\ 1 & x_2 & x_2^2 & \cdots & x_2^{n-1} \\ \vdots & & & & \\ 1 & x_n & x_n^2 & \cdots & x_n^{n-1} \end{bmatrix}_{n \times n}$

where $x_1, \dots, x_n$ are real numbers.

"product" instead of "sum"

Show that $\det(V) = \prod_{1 \le i < j \le n} (x_j - x_i)$.

[Hint: start with a 3x3 matrix and show $\det(V) = (x_3 - x_2)(x_3 - x_1)(x_2 - x_1)$.]

EXERCISE

Complete the Chapter 5 conceptual questions. You can find them under "Other Materials" on the course website. Watch the Chapter 5 videos as well. (Also under "Other Materials.")