Lecture #11

4.3 Row and Column Spaces

Recall from 4.2 that we developed techniques for determining a basis of a subspace when the subspace is described as the span of a set of vectors. In fact, we had two techniques: ① make the set of vectors into a matrix by inserting each vector as a row into a matrix, and ② make the vectors into a matrix by inserting each vector as a column into a matrix.

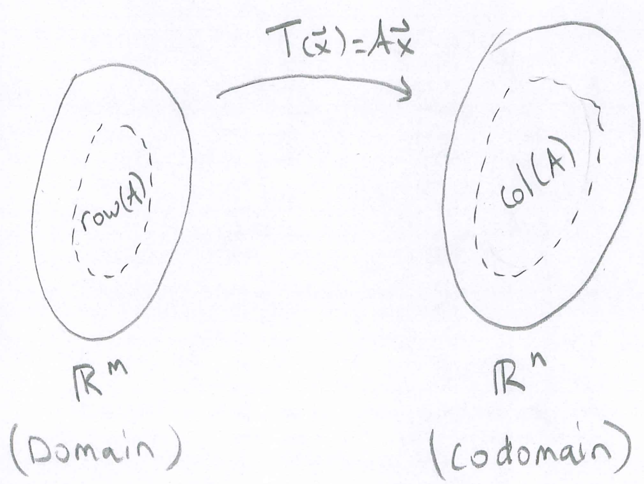
Now, instead of being given a set of vectors, what if we were given a matrix? A matrix has both row vectors and column vectors ... and we can consider the span of each set separately. How do these sets of vectors compare? What is the dimension of the span of each set? (How does it compare?) We'll answer these questions here, and discover a fundamental property of matrices (and linear transformations) that is called the Rank-nullity Theorem. First, we will formalize some of this by making a few definitions.

DEF Let A be an $n \times m$ matrix.

(a) The $\boxed{\text{row space}}$ of A is the subspace of $\mathbb{R}^m$ spanned by the row vectors of A and is denoted $\boxed{\text{row}(A).}$

(b) The $\boxed{\text{column space}}$ of A is the subspace of $\mathbb{R}^n$ spanned by the column vectors of A and is denoted by $\boxed{\text{col}(A).}$

REMARK Let $T : \mathbb{R}^m \to \mathbb{R}^n$ be the linear transformation such that $T(\vec{x}) = A\vec{x}$, A $n \times m$ matrix. Notice that the subspace row(A) is in $\mathbb{R}^m$, the domain, and the subspace col(A) is in $\mathbb{R}^n$, the codomain.



Notice, also, col(A) = range(T) ! ← look back at the definition to confirm!

Next, recall Method 1 and Method 2 from 4.2. We are going to re-cast the results of those recipes in terms of our new definitions:

THM Let A be a matrix and B an equivalent matrix in echelon form. Then

1. The nonzero rows of B form a basis for $\text{row}(A)$.
2. The columns of A corresponding to the pivot columns of B form a basis for $\text{col}(A)$.

Next, let's think about how ① and ② relate to each other ... this will lead us to understanding the first key connection between $\text{row}(A)$ and $\text{col}(A)$.

(*) **THM** For any matrix A, the dimension of the row space is equal to the dimension of the column space. (*)

Why is this true? What does this have to do with ① and ② above? Think about it like this. Given a matrix A, if we put the matrix in echelon form (call the new matrix B), then we know two things. First, the non-zero rows of B form a basis for row(A), meaning the dimension of row(A) is the <u>number</u> of non-zero rows in B.

Next, we know that the columns of A corresponding to the pivot columns of B form a basis for col(A). In other words, the dimension of col(A) is the <u>number</u> of pivot columns in B.

Now, how does the number of pivot columns relate to the number of non-zero rows in B? Recall from <u>EXAMPLE 4</u> in <u>Lecture #10</u> (4.2):

$$A := \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 1 & 2 & 3 \\ 1 & 3 & 5 & 7 \end{bmatrix} \sim \begin{array}{rc} \text{non-zero rows} \rightarrow & \left[\begin{array}{cccc} 1 & 2 & 3 & 4 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 0 \end{array}\right] \\ & \begin{array}{cccc} \uparrow & \uparrow & & \end{array} \\ & \text{pivot columns} \end{array} =: B$$

Every non-zero row in B has a leading term, or a <u>pivot</u>. Each <u>pivot</u> is in a pivot column. In other words, the number of non-zero rows in B is the same as the number of pivot columns.

Next, recall a definition from 4.2, nullity. The nullity of a matrix A is the dimension of the null space, denoted $\boxed{\text{nullity}(A)}$. Here is a similar, related definition:

DEF The $\boxed{\text{rank}}$ of a matrix A is the dimension of the column space (or row space, since they are the same) of A, and is denoted $\boxed{\text{rank}(A)}$.

EXAMPLE 1

Find the rank and nullity for the matrix

$$A = \begin{bmatrix} 1 & -2 & 3 & 0 & -1 \\ 2 & -4 & 7 & -3 & 3 \\ 3 & -6 & 8 & 3 & -8 \end{bmatrix}.$$

First, notice that if we put A in echelon form, we can quickly compute $\text{rank}(A)$:

non-zero rows

$$A = \begin{bmatrix} 1 & -2 & 3 & 0 & -1 \\ 2 & -4 & 7 & -3 & 3 \\ 3 & -6 & 8 & 3 & -8 \end{bmatrix} \overset{\text{(check!)}}{\sim} \begin{bmatrix} 1 & -2 & 3 & 0 & -1 \\ 0 & 0 & 1 & -3 & 5 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} =: B$$

$$\begin{array}{ccccc} \uparrow & & \uparrow & & \\ & & & & \end{array}$$

pivot columns

So we see $\underline{\text{rank}(A) = 2}$. To compute nullity$(A)$, we have to consider $A\vec{x} = \vec{0}$, and compute the dimension of the solution set. We need an augmented matrix to do this:

$$\left[\begin{array}{ccccc|c} 1 & -2 & 3 & 0 & -1 & 0 \\ 2 & -4 & 7 & -3 & 3 & 0 \\ 3 & -6 & 8 & 3 & -8 & 0 \end{array}\right] \overset{\text{same row operations as before!}}{\sim} \left[\begin{array}{ccccc|c} 1 & -2 & 3 & 0 & -1 & 0 \\ 0 & 0 & 1 & -3 & 5 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

Now, notice that x_2, x_4, and x_5 are <u>free</u> <u>variables</u> (these are exactly corresponding to the columns that are <u>not</u> pivot columns!).

So, let $x_2 = r$, $x_4 = s$, $x_5 = t$, and back substitute:

2nd row in matrix (echelon form) $\Rightarrow \quad x_3 - 3x_4 + 5x_5 = 0 \quad \rightsquigarrow \quad x_3 = \underbrace{3s - 5t}$

1st row in matrix $\Rightarrow \quad x_1 - 2x_2 + 3x_3 - x_5 = 0 \quad \rightsquigarrow \quad x_1 = 2r - 3x_3 + t$

$$= 2r - 9s + 16t$$

So our solution is

$$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} 2r - 9s + 16t \\ r \\ 3s - 5t \\ s \\ t \end{bmatrix} = r\begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + s\begin{bmatrix} -9 \\ 0 \\ 3 \\ 1 \\ 0 \end{bmatrix} + t\begin{bmatrix} 16 \\ 0 \\ -5 \\ 0 \\ 1 \end{bmatrix}$$

and we see that $\mathrm{null}(A) = \mathrm{span}\left\{ \begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -9 \\ 0 \\ 3 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 16 \\ 0 \\ -5 \\ 0 \\ 1 \end{bmatrix} \right\}$, and since these are necessarily linearly independent (why?), we see that nullity$(A) = 3$.

∎

Notice ... in the last example, the number of pivot columns in B was rank(A), and the number of free variables was nullity(A). It always works like this! Notice:

$$A = \begin{bmatrix} 1 & -2 & 3 & 0 & -1 \\ 2 & -4 & 7 & -3 & 3 \\ 3 & -6 & 8 & 3 & -8 \end{bmatrix} \sim \begin{array}{c} \begin{bmatrix} 1 & -2 & 3 & 0 & -1 \\ 0 & 0 & 1 & -3 & 5 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \\ \begin{array}{ccccc} \text{pivot column} & \text{free variable} & \text{pivot column} & \text{free variable} & \text{free variable} \end{array} \end{array} \leftarrow \text{echelon form}$$

# of Pivot columns + # of Free Variables = # of columns!

This is a theorem:

(*)

THM (Rank-Nullity Theorem)

Let A be an $n \times m$ matrix. Then

$$\text{rank}(A) + \text{nullity}(A) = m.$$

(*)

Here's a fun exercise that goes along with the rank-nullity theorem:

EXERCISE

Find a linear transformation T that has kernel equal to $\text{span}\{\vec{v}_1, \vec{v}_2\}$, where

$$\vec{v}_1 = \begin{bmatrix} 1 \\ 0 \\ -2 \\ 1 \end{bmatrix} \quad \text{and} \quad \vec{v}_2 = \begin{bmatrix} 0 \\ 1 \\ 3 \\ 2 \end{bmatrix}.$$

[HINT: You want to find a matrix A such that $T(\vec{x}) = A\vec{x}$, and $\ker(T) = \text{null}(A) = \text{span}\{\vec{v}_1, \vec{v}_2\}$. Notice $\vec{v}_1$ and $\vec{v}_2$ are linearly independent, so $\text{nullity}(A) = 2$. Since

$$A\vec{v}_1 = \vec{0}$$

we see that A must have 4 columns, and by the rank nullity theorem, $\text{rank}(A) = 2$. Thus, A must have 2 or more rows. Start with the assumption that A is 2×4 and see if you can find the transformation.]

Next, we'll update our big theorem by making the following observation: if the span of a set of column vectors in an $n \times n$ matrix A is $\mathbb{R}^n$, then $\text{col}(A) = \mathbb{R}^n$. Since the dimension of $\text{col}(A)$ = dimension of $\text{row}(A)$, $\text{row}(A) = \mathbb{R}^n$, as well.

Since the dimension of $\mathrm{col}(A)$ is n, $\mathrm{rank}(A) = n$. Then, by the rank-nullity theorem, $\mathrm{nullity}(A) = 0$, in other words $\mathrm{null}(A) = \{\vec{0}\}$, (But we do not need to add this last statement to our theorem. Check out (g).)

THM (The Unifying Theorem, vers. 6)

Let $S = \{\vec{a}_1, \ldots, \vec{a}_n\}$ be a set of n vectors in $\mathbb{R}^n$, and let $A = [\vec{a}_1 \cdots \vec{a}_n]$, and let $T: \mathbb{R}^n \to \mathbb{R}^n$ be given by $T(\vec{x}) = A\vec{x}$.

Then the following are equivalent:

(a) S spans $\mathbb{R}^n$.

(b) S is linearly independent.

(c) $A\vec{x} = \vec{b}$ has a unique solution for all $\vec{b}$ in $\mathbb{R}^n$.

(d) T is onto.

(e) T is one-to-one.

(f) A is invertible.

(g) $\ker(T) = \{\vec{0}\}$.

(h) S is a basis for $\mathbb{R}^n$

(i) $\mathrm{col}(A) = \mathbb{R}^n$

(j) $\mathrm{row}(A) = \mathbb{R}^n$

(k) $\mathrm{rank}(A) = n$.