4.2 Basis and Dimension

Recall our unifying theorem for a moment. In $\mathbb{R}^n$, if we have n vectors that are linearly independent, then these vectors span $\mathbb{R}^n$, and there is a unique solution $\vec{x}$ to the equation $A\vec{x} = \vec{b}$ for every $\vec{b}$ in $\mathbb{R}^n$. Notice, we can re-phrase this as:

$$A\vec{x} = \vec{b}, \quad A = [\vec{a}_1 \cdots \vec{a}_n]_{n \times n}$$

$$x_1\vec{a}_1 + x_2\vec{a}_2 + \cdots + x_n\vec{a}_n = \vec{b}.$$

$\vec{a}_1, \ldots, \vec{a}_n$ are vectors in $\mathbb{R}^n$, and for any vector $\vec{b}$, we can find "coordinates $(x_1, \ldots, x_n)$" describing $\vec{b}$ in terms of the set of vectors $\{\vec{a}_1, \ldots, \vec{a}_n\}$. This means that every set of "coordinates" describes a unique vector $\vec{b}$, and all vectors $\vec{b}$ in $\mathbb{R}^n$ can be described this way.

not being rigorous!

EXERCISE Let $\{\vec{e}_1, \ldots, \vec{e}_n\}$ be a set of vectors in $\mathbb{R}^n$. Does this set have the property described above? Now, return to the proof of the Theorem in Lecture #6 (3.1), p. 12, and look at the property of the set of vectors in the proof that we used.

It turns out that $\{\vec{a}_1, \ldots, \vec{a}_n\}$ is an example of a basis, something that enables this kind of description.

Furthermore, we can define a basis not only in $\mathbb{R}^n$ for any n, but for any subspace of $\mathbb{R}^n$. (Careful though... we'll see that one subspace does not have a basis.)

DEF A set $\mathcal{B} = \{\vec{u}_1, .., \vec{u}_m\}$ is a $\boxed{\text{basis}}$ for a subspace S if

① $\mathcal{B}$ spans S.

② $\mathcal{B}$ is linearly independent.

EXAMPLE 1

$\{\vec{e}_1, .., \vec{e}_n\}$ is a basis for $\mathbb{R}^n$.

To show this, notice we have n-vectors in $\mathbb{R}^n$, so conditions ① and ② above are equivalent. In other words, we only need to show that one of them holds. Lets show the set is linearly independent:

$$\left[\begin{array}{cccc|c} \vec{e}_1 & \vec{e}_2 & \cdots & \vec{e}_n & \begin{matrix} 0 \\ 0 \\ \vdots \\ 0 \end{matrix} \end{array}\right] = \left[\begin{array}{cccc|c} 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & & \vdots \\ 0 & 0 & & 1 & 0 \end{array}\right],$$

and we see there is only one solution (the trivial solution), so $\{\vec{e}_1, .., \vec{e}_n\}$ is linearly independent. Thus, it is a basis.

EXERCISE Use the definition of span to show $\{\vec{e}_1, .., \vec{e}_n\}$ spans $\mathbb{R}^n$.

EXAMPLE 2

Show that $\{\vec{0}\}$ in $\mathbb{R}^n$ has no basis.
(zero subspace)

Notice, the only set of vectors we can make is $\{\vec{0}\}$, but this set is (weirdly) not linearly independent:

$$r\vec{0} = \vec{0}$$

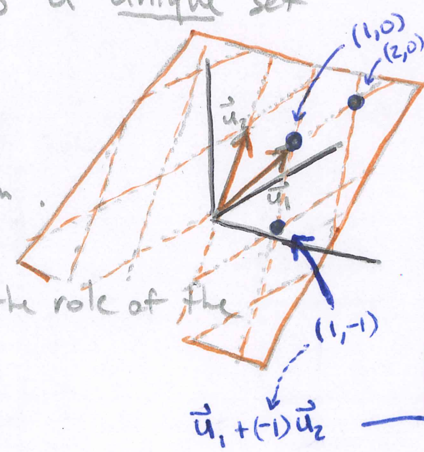
for any r a real number. (Do you see how this breaks linear independence?) Thus, since $\{\vec{0}\}$ is not a basis for the zero subspace, there is no basis.

Now, let's give a theorem, and notice the theorem matches our motivation for defining a basis ... (i.e. we have the right definition to match that intuition!)

THM Let $\mathcal{B} = \{\vec{u}_1, \ldots, \vec{u}_m\}$ be a basis for a subspace S. For every vector $\vec{s}$ in S, there exists a unique set of scalars $s_1, \ldots, s_m$ such that

$$\vec{s} = s_1\vec{u}_1 + \cdots + s_m\vec{u}_m.$$

(The scalars $s_1, \ldots, s_m$ are filling the role of the "coordinates" on the first page.)



THINK: "A basis enables us to express every vector in a subspace in exactly one unique way."

If you are interested in seeing a proof of this theorem, check out p. 179 in the textbook.

Here, however, we will give one other theorem, and then use it to do a few example problems.

<u>THM</u> Let A and B be equivalent matrices. Then the subspace spanned by the rows of A is the same as the subspace spanned by the rows of B.

<u>EXERCISE</u> How would you go about convincing yourself that the theorem above is true? Remember, A and B being equivalent means that there are a set of row operations you can do to A to make it B. If you do one row operation on A, will the subspace spanned by the rows change?

<u>REMARK</u> We have yet to talk about "span of the row vectors" of a matrix. We call this the row space, and the row space is a subspace of the <u>domain</u> of the associated linear transformation. Similarly, the "span of the column vectors" is called the column space, and is a subspace of the <u>codomain</u> of the associated linear transformation. These space are the topic of 4.3 (the next set of lecture notes!).

We are going to use this theorem to help us do an example:

EXAMPLE 3 (Method 1)

Let S be the subspace of $\mathbb{R}^4$ spanned by the vectors

$$\vec{u}_1 = \begin{bmatrix} -1 \\ 2 \\ 3 \\ 1 \end{bmatrix}, \quad \vec{u}_2 = \begin{bmatrix} -6 \\ 7 \\ 5 \\ 2 \end{bmatrix}, \quad \vec{u}_3 = \begin{bmatrix} 4 \\ -3 \\ 1 \\ 0 \end{bmatrix}.$$

Find a basis for S.

Let's start with a simple question: is $\{\vec{u}_1, \vec{u}_2, \vec{u}_3\}$ a basis? To be a basis, it would have to be linearly independent. We could check for linear independence ... but would that tell us what vector to remove? (It does ... this will be method 2 .. but for this example, we are going to use the theorem. It turns out that method 2 has a subtlety ..)

Since two equivalent matrices have the same "row space", let's put $\vec{u}_1, \vec{u}_2, \vec{u}_3$ into a matrix as rows, do Gaussian elimination, and see if one of the rows goes away (this would mean one of the rows was a linear combination of the others! Think through that ...)

$$A = \begin{bmatrix} -1 & 2 & 3 & 1 \\ -6 & 7 & 5 & 2 \\ 4 & -3 & 1 & 0 \end{bmatrix} \begin{array}{l} \vec{u}_1 \\ \\ \\ \end{array}$$

Doing Gaussian elimination, we can put A in echelon form:

$$A = \begin{bmatrix} -1 & 2 & 3 & 1 \\ -6 & 7 & 5 & 2 \\ 4 & -3 & 1 & 0 \end{bmatrix} \underset{\text{(check!)}}{\sim} \begin{bmatrix} -1 & 2 & 3 & 1 \\ 0 & 5 & 13 & 4 \\ 0 & 0 & 0 & 0 \end{bmatrix} =: B$$

Then $A \sim B$, and by the theorem, we know the span of the rows of A is the same as the span of the rows of B. Next, notice that since B is in echelon form, the non-zero rows are linearly independent.

<u>EXERCISE</u> Can you see why? Give a brief argument just looking at the vectors. Can you make an argument for any set of non-zero row vectors in an echelon form matrix?

Then:

$$\text{span}\left\{ \begin{bmatrix} -1 \\ 2 \\ 3 \\ 1 \end{bmatrix}, \begin{bmatrix} -6 \\ 7 \\ 5 \\ 2 \end{bmatrix}, \begin{bmatrix} 4 \\ -3 \\ 1 \\ 0 \end{bmatrix} \right\} = \text{span}\left\{ \begin{bmatrix} -1 \\ 2 \\ 3 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 5 \\ 13 \\ 4 \end{bmatrix} \right\}$$

Since the set on the right is linearly independent and spans the subspace, it is a basis for S.

<u>EXERCISE</u> Based on the row operations we have to do to turn A into B, can you tell which vector in A was in the span of the other two? That vector is "redundant"... so you could just remove that from the set to get another basis. But, then you would have to show the remaining vectors are linearly independent.

Now, let's go back to that other idea, just checking for linear independence. That method involves thinking of the set of vectors as column vectors in a matrix, and solving the homogeneous system:

$$U = [\vec{u}_1 \ \vec{u}_2 \ \vec{u}_3], \quad U\vec{x} = \vec{0}.$$

This would tell us whether or not the set is linearly independent... but it does not tell us where the dependencies are.

We have a theorem that could help us fix this problem:

<u>THM</u> Suppose $U = [\vec{u}_1 \cdots \vec{u}_m]$ and $V = [\vec{v}_1 \cdots \vec{v}_m]$ are two equivalent matrices. Then any linear dependence that exists among the vectors $\vec{u}_1, \dots, \vec{u}_m$ also exists among $\vec{v}_1, \dots, \vec{v}_m$.

In other words, if $2\vec{v}_1 + 3\vec{v}_2 - \vec{v}_4 = \vec{v}_5$, then $2\vec{u}_1 + 3\vec{u}_2 - \vec{u}_4 = \vec{u}_5$. (!!) So, if we can take a matrix with column vectors, do Gaussian elimination and identify dependency in the echelon form matrix, then we understand dependency in the original set of vectors.

<u>EXERCISE</u> Can you see why this theorem would be true? What if you perform <u>one</u> row operation on a matrix? What is happening to the column vectors?

Let's try an example.

EXAMPLE 3 (Method 2)

Repeat example 3, but set up a matrix with column vectors $\vec{u}_1, \vec{u}_2$, and $\vec{u}_3$ instead of row vectors.

$$A := \begin{bmatrix} \vec{u}_1 & \vec{u}_2 & \vec{u}_3 \end{bmatrix} = \begin{bmatrix} -1 & -6 & 4 \\ 2 & 7 & -3 \\ 3 & 5 & 1 \\ 1 & 2 & 0 \end{bmatrix}$$

Perform Gaussian Elimination to put the matrix in echelon form:

$$A = \begin{bmatrix} -1 & -6 & 4 \\ 2 & 7 & -3 \\ 3 & 5 & 1 \\ 1 & 2 & 0 \end{bmatrix} \underset{\text{(check!)}}{\sim} \begin{bmatrix} 1 & 6 & -4 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} =: B$$

Notice, in matrix B, we can see that

$$\boxed{(-1)\cdot(2^{nd} \text{ column}) + (2)\cdot(1^{st} \text{ column}) = 3^{rd} \text{ column},}$$

so we have found a dependency! By the theorem, the dependency holds for matrix A, namely

$$-\vec{u}_2 + 2\vec{u}_1 = \vec{u}_3 \qquad \text{(check!)}$$

Thus, $\vec{u}_3$ is a redundant vector, so

$$\text{Span}\{\vec{u}_1, \vec{u}_2, \vec{u}_3\} = \text{Span}\{\vec{u}_1, \vec{u}_2\}.$$

But that's not all, our theorem tells us more. Notice that in B, the first column and second column form a linearly independent set. Since there are no dependencies between the first column and second column in B, there cannot be any dependencies between the first and second column in A! Thus, $\vec{u}_1$ and $\vec{u}_2$ are also linearly independent. Then

$$\mathcal{B} = \{ \vec{u}_1 , \vec{u}_2 \}$$

is a basis for S.

$\square$

Next, notice that the first and second columns of matrix B in the last example are pivot columns!

$$B = \begin{bmatrix} 1 & 6 & -4 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

pivot columns!

And notice that the pivot columns are linearly independent! We can use this as a shortcut to claim $\{\vec{u}_1, \vec{u}_2\}$ is a basis. In some sense, method 2 reduces to identifying pivot columns in an echelon form matrix, then using those columns in the original matrix.

Now, notice that the last two examples give us two procedures for generating a basis of a subspace.

Row Method

Method 1 (To find a basis for span $\{\vec{u}_1, \ldots, \vec{u}_m\}$.)

① Use the vectors $\vec{u}_1, \vec{u}_2, \ldots, \vec{u}_m$ to form the rows of a matrix A.

② Transform A to echelon form, call the new matrix B.

③ The non-zero rows of B give a basis for span$\{\vec{u}_1, \ldots, \vec{u}_m\}$.

Column Method

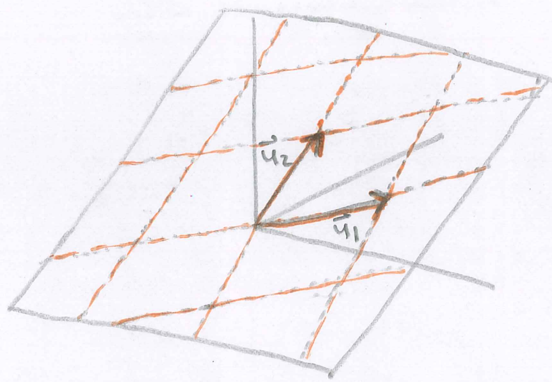
Method 2 (To find a basis for span $\{\vec{u}_1, \ldots, \vec{u}_m\}$.)

① Use the vectors $\vec{u}_1, \ldots, \vec{u}_m$ to form the columns of a matrix A.

② Transform A to echelon form, call the new matrix B. The pivot columns of B will be linearly independent, and the other columns will be linearly dependent on the pivot columns.

③ The columns of A corresponding to the pivot columns of B form a basis for span$\{\vec{u}_1, \ldots, \vec{u}_m\}$.

Next, let's lean on our intuition for a minute: is it possible for a subspace to have a basis with three vectors, and then a different basis with only two vectors? No! For example, if we have a plane passing through the origin in $\mathbb{R}^3$ with a basis $\mathcal{B} = \{\vec{u}_1, \vec{u}_2\}$:



Can we add a third vector? No! Then the set would be linearly dependent. If we tried to remove a vector... the set wouldn't span the plain. In fact, If we choose any other set of vectors in the plane, we won't span it unless there are at least two vectors in the set. Similarly, the set won't be linearly independent unless there are two or fewer vectors in the set. So we must have exactly two to make a basis. This is true in any subspace - well, not that we would need two vectors - but that all basis' for the subspace have the <u>same</u> <u>number</u> <u>of</u> <u>vectors</u>!

THM If S is a subspace of $\mathbb{R}^n$, then every basis of S has the same number of vectors.

This enables us to make a definition of a term we have used loosely thus far:

DEF Let S be a subspace of $\mathbb{R}^n$. Then the $\boxed{\text{dimension}}$ of S is the number of vectors in any basis.

EXAMPLE 4

Let $S = \text{span}\left\{ \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 1 \\ 3 \\ 5 \\ 7 \end{bmatrix} \right\}$. What is the dimension of S?

Let's generate a basis ... and count the number of elements. We'll use method 1 to do this (row vectors):

$$\begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 1 & 2 & 3 \\ 1 & 3 & 5 & 7 \end{bmatrix} \overset{\text{(check!)}}{\sim} \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

So we see the matrix in echelon form has two non-zero rows.

Thus, $\left\{ \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 2 \\ 3 \end{bmatrix} \right\}$ is a basis, and the dimension of S is two.

(For another example, using Method 2, see Example 3 in the text.)

Note, we can give one more theorem, slightly generalizing one of the points made two pages back:

THM If $\mathcal{U} = \{\vec{u}_1, \vec{u}_2, \ldots, \vec{u}_m\}$ be a set of vectors in a subspace $S \neq \{\vec{0}\}$ of $\mathbb{R}^n$.

① If $\mathcal{U}$ is linearly independent, either $\mathcal{U}$ is a basis for S <u>or</u> you can add vectors to form a basis.

② If $\mathcal{U}$ spans S, then either $\mathcal{U}$ is a basis for S <u>or</u> you can remove vectors from $\mathcal{U}$ to form a basis.

EXAMPLES

Expand $\mathcal{U} = \left\{ \begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix}, \begin{bmatrix} 3 \\ 2 \\ 4 \end{bmatrix} \right\}$ to be a basis for $\mathbb{R}^3$.

NOTE $\mathcal{U}$ does not span $\mathbb{R}^3$ (how do we know this?)

First, let's make sure $\mathcal{U}$ is a linearly independent set, and if it is, since we need three vectors to span $\mathbb{R}^3$, we can add one to make a basis.

$$\left[\begin{array}{cc|c} 1 & 3 & 0 \\ 1 & 2 & 0 \\ -2 & 4 & 0 \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & 3 & 0 \\ 0 & -1 & 0 \\ 0 & 10 & 0 \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & 3 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

$\Rightarrow$ one solution (trivial)

$\Rightarrow$ the set is linearly independent.

Great. So now we just need to add a vector ... but what vector do we add? It's a little hard to just pick one, so instead ... remember that we already have a basis for $\mathbb{R}^3$, namely $\{\vec{e}_1, \vec{e}_2, \vec{e}_3\}$. If we add all three of those vectors to our set, our set will certainly span $\mathbb{R}^3$, but then we'll have too many vectors. We only really need to add one of these vectors ... but which one? We could add them one at a time, testing for linear independence, and as soon as we see a set that is linearly independent, that will be a basis.

However, we can also try to generate a basis from the set $\left\{ \begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix}, \begin{bmatrix} 3 \\ 2 \\ 4 \end{bmatrix}, \vec{e}_1, \vec{e}_2, \vec{e}_3 \right\}$ since

$$\operatorname{span}\left\{ \begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix}, \begin{bmatrix} 3 \\ 2 \\ 4 \end{bmatrix}, \vec{e}_1, \vec{e}_2, \vec{e}_3 \right\} = \mathbb{R}^3.$$

Let's try that, using method 2 this time (column vectors). We'll try to make the first two columns pivot columns ... (why?) (they will be!)

$$A = \begin{bmatrix} 1 & 3 & 1 & 0 & 0 \\ 1 & 2 & 0 & 1 & 0 \\ -2 & -4 & 0 & 0 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 3 & 1 & 0 & 0 \\ 0 & 1 & 1 & -1 & 0 \\ 0 & 0 & 0 & 2 & 1 \end{bmatrix}$$

(check!)

these vectors generate a basis!

so we need

pivot columns

So, a basis for $\mathbb{R}^3$ is:

$$\left\{ \begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix}, \begin{bmatrix} 3 \\ 2 \\ 4 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \right\}.$$

$\boxtimes$

Next, let's recall an interesting subspace, the <u>null space</u> of a matrix. Recall, the null space of a matrix A, denoted $\text{null}(A)$, is the solution set to $A\vec{x} = \vec{0}$. (i.e. Everything that A sends to $\vec{0}$!)

<u>DEF</u> The $\boxed{\text{nullity}}$ of a matrix A is the dimension of the null space.

It turns out that the nullity of a matrix is a very important number (which will come up again in the next section). If you want to see an example of a basis type question with the null space involved, try the following

<u>EXERCISE</u> Check out Example 5 on page 184 of the text.

Ultimately, this is very similar to the last example we did!

Next, let's state a fact ... (We knew this was true in $\mathbb{R}^n$, but now we can claim it for any subspace!)

THM Let $\mathcal{U} = \{\vec{u}_1, \ldots, \vec{u}_m\}$ be a set of m vectors in a space S of dimension m. If $\mathcal{U}$ is either linearly independent or spans S, then $\mathcal{U}$ is a basis for S.

And now, we can update our unifying theorem!

THM (Unifying Theorem, vers. 5)
Let $S = \{\vec{a}_1, \ldots, \vec{a}_n\}$ be a set of n vectors in $\mathbb{R}^n$, let $A = [\vec{a}_1 \cdots \vec{a}_n]$, and let $T: \mathbb{R}^n \to \mathbb{R}^n$ be $T(\vec{x}) = A\vec{x}$. Then the following are equivalent:

(a) S spans $\mathbb{R}^n$.
(b) S is linearly independent.
(c) $A\vec{x} = \vec{b}$ has a unique solution for all $\vec{b}$ in $\mathbb{R}^n$.
(d) T is onto.
(e) T is one-to-one.
(f) A is invertible.
(g) $\ker(T) = \{\vec{0}\}$.
(h) S is a basis for $\mathbb{R}^n$.

one more page!
↓

(A nifty) **EXERCISE** Let $x_1, x_2, \ldots, x_n$ be real numbers.

this is hard!

Let

$$V = \begin{bmatrix} 1 & x_1 & x_1^2 & \cdots & x_1^{n-1} \\ 1 & x_2 & x_2^2 & \cdots & x_2^{n-1} \\ 1 & x_3 & x_3^2 & \cdots & x_3^{n-1} \\ & \vdots & \vdots & & \vdots \\ 1 & x_n & x_n^2 & \cdots & x_n^{n-1} \end{bmatrix}_{n \times n}$$

Show that if $x_1, x_2, \ldots, x_n$ are distinct, then the columns of V form a basis for $\mathbb{R}^n$.

[Hint: It's a square matrix ... use the unifying theorem. Can you show the columns are linearly independent? Try generating a system of equations from the homogeneous equation $V\vec{y} = \vec{0}$.]

(don't use $\vec{x}$ since we are already using lots of x's!)

[Hint 2: The solution ... if you get stuck .. is on p. 186/187 of the textbook.]