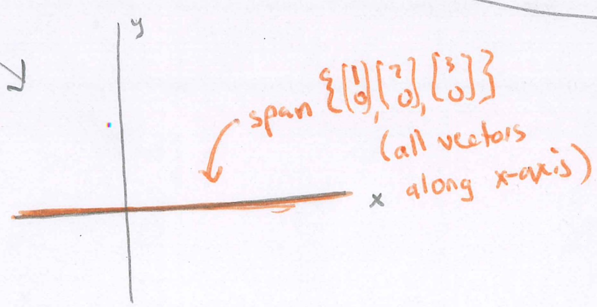
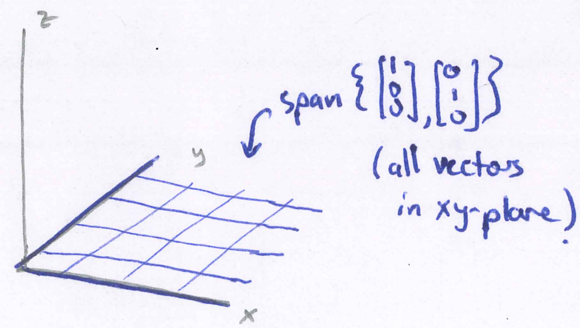
4.1 Subspaces

Recall that if we take the span of a set of vectors in $\mathbb{R}^n$, it may or may not end up being all of $\mathbb{R}^n$. For example, in $\mathbb{R}^2$, $\boxed{\text{span}\left\{\begin{bmatrix}1\\0\end{bmatrix}, \begin{bmatrix}2\\0\end{bmatrix}, \begin{bmatrix}3\\0\end{bmatrix}\right\}}$ is **not** $\mathbb{R}^2$. But, as we noticed when we thought about span geometrically, it is a line, and more specifically, a line passing through the origin. Similarly, in three dimensions, $\boxed{\text{span}\left\{\begin{bmatrix}1\\0\\0\end{bmatrix}, \begin{bmatrix}0\\1\\0\end{bmatrix}\right\} \neq \mathbb{R}^3}$, but we know it's a plane that passes through the origin (in fact, it's just the xy-plane in $\mathbb{R}^3$!).





It turns out that these "spaces" are examples of *subspaces*, special spaces which satisfy properties that will make them structurally important... i.e. they are *almost* like "$\mathbb{R}$" or "$\mathbb{R}^2$", but *are not* the same as $\mathbb{R}$ or $\mathbb{R}^2$. (In fact, a common exam question asks "Is $\mathbb{R}^2$ a subspace of $\mathbb{R}^3$?"... the answer, as we will see, is an *emphatic* no!)

Let's be more rigorous now. Here's a definition:

DEF A subset of $\mathbb{R}^n$ is a $\boxed{\text{subspace}}$ if S satisfies the three following conditions:

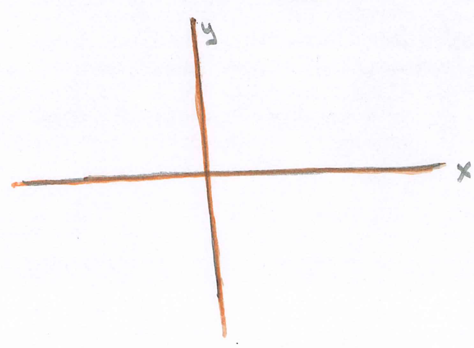
1. S contains $\vec{0}$, the zero vector.
2. If $\vec{u}$ and $\vec{v}$ are in S, then $\vec{u}+\vec{v}$ is in S.
3. If r is a real number and $\vec{u}$ is in S, then $r\vec{u}$ is also in S.

EXERCISE Check that our first two examples are, in fact, subspaces.

EXERCISE
- Is the span of a set of vectors always a subspace?
- (*) Is the solution (set) to an equation $A\vec{x}=\vec{b}$ for any $\vec{b}$ a subspace? What if $\vec{b}=\vec{0}$?

EXAMPLE 1

Let S be the subset of $\mathbb{R}^2$ consisting of the x-axis and y-axis. Show that S is not a subspace.



← **NOTE** Vectors off the two axes are not in S.

First, notice $\vec{0}$ is in S, so condition ① holds. Let's try to break condition ②. Notice $\begin{bmatrix}1\\0\end{bmatrix}$ and $\begin{bmatrix}0\\1\end{bmatrix}$ are in S, but

$$\begin{bmatrix}1\\0\end{bmatrix} + \begin{bmatrix}0\\1\end{bmatrix} = \begin{bmatrix}1\\1\end{bmatrix}$$

is <u>not</u> in S (it doesn't point along an x-axis). This breaks condition ②, so we see S is not a subspace.

<u>EXERCISE</u> Show that the subset S of $\mathbb{R}^2$ consisting of all vectors of the form $\begin{bmatrix}a\\b\end{bmatrix}$, where a,b are integers, is <u>not</u> a subspace of $\mathbb{R}^2$.

(HINT: Conditions ① and ② hold. You can break condition ③.)

Did you do the exercises on the last page? If not, the next theorem might spoil some of the fun...

<u>THM</u> Let $S = \text{span}\{\vec{u}_1, \vec{u}_2, \ldots, \vec{u}_m\}$ be in $\mathbb{R}^n$. Then S is a subspace!

<u>DEF</u> If we see that a subspace S is $\text{span}\{\vec{u}_1, \ldots, \vec{u}_m\}$, we often say S is the **subspace spanned** by $\{\vec{u}_1, \ldots, \vec{u}_m\}$ or that S is the **subspace generated** by $\{\vec{u}_1, \ldots, \vec{u}_m\}$.

pf: Recall span $\{\vec{u}_1, \dots, \vec{u}_m\}$ is the set of all linear combinations of $\{\vec{u}_1, \dots, \vec{u}_m\}$. Let's make sure the conditions all hold:

① $\vec{0}$ is in span $\{\vec{u}_1, \dots, \vec{u}_m\}$:

$$x_1 \vec{u}_1 + x_2 \vec{u}_2 + x_3 \vec{u}_3 + \dots + x_m \vec{u}_m \in \text{span}\{\vec{u}_1, \dots, \vec{u}_m\}$$

pick $x_1 = x_2 = x_3 = \dots = x_m = 0$, then $\begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix} \in \text{span}\{\vec{u}_1, \dots, \vec{u}_m\}$.

(Alternatively, solve

$$x_1 \vec{u}_1 + \dots + x_m \vec{u}_m = \vec{0}.$$

Notice: always one solution! This is just a homogeneous system.)

② Let $\vec{u}, \vec{v} \in \text{span}\{\vec{u}_1, \dots, \vec{u}_m\}$. Need to show $\vec{u}+\vec{v} \in \text{span}\{\vec{u}_1, \dots, \vec{u}_m\}$.

$\vec{u} = c_1 \vec{u}_1 + \dots + c_m \vec{u}_m$ for some $c_1, \dots, c_m$, real numbers.

$\vec{v} = d_1 \vec{u}_1 + \dots + d_m \vec{u}_m$ for some $d_1, \dots, d_m$, real numbers.

Then,

$$\vec{u}+\vec{v} = (c_1+d_1)\vec{u}_1 + \dots + (c_m+d_m)\vec{u}_m,$$

and since $c_1+d_1, \dots, c_m+d_m$ are just scalars,

$\vec{u}+\vec{v} \in \text{span}\{\vec{u}_1, \dots, \vec{u}_m\}$.

③ Let r be a real number, let $\vec{u} \in \text{span}\{\vec{u}_1, \dots, \vec{u}_m\}$.

Then

$\vec{u} = c_1 \vec{u}_1 + \dots + c_m \vec{u}_m$ for some $c_1, \dots, c_m$ real numbers.

Then

$$\begin{aligned} r\vec{u} &= r(c_1 \vec{u}_1 + \dots + c_m \vec{u}_m) \\ &= rc_1 \vec{u}_1 + \dots + rc_m \vec{u}_m \end{aligned}$$

and since $rc_1, rc_2, \dots, rc_m$ are just scalars,

$r\vec{u} \in \text{span}\{\vec{u}_1, \dots, \vec{u}_m\}$ as needed. $\boxtimes$

Now, we are at a point where we can generate a recipe for how to quickly identify whether or not a subset is a subspace:

RECIPE

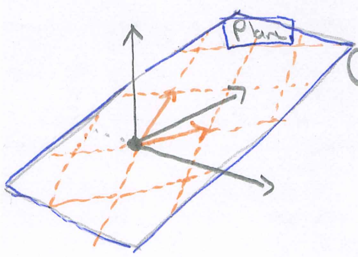
1. First, make sure $\vec{0}$ is in S, otherwise it is definitely not a subspace!

2. Next, if you can show S is generated by a set of vectors (i.e. is the span of a set of vectors), then by the THM, stated 2 pages back, S is a subspace.

3. If it is not clear that S is generated by a set of vectors, then check conditions 2 and 3 by hand.

BUT! Be careful not to lean on this recipe too much. It is always nice to have a recipe, but it is often better to have good intuition over recipes. You will definitely need to use the recipe to identify conclusively whether or not some subset S is a subspace .. but:

EXERCISE 1 Show that all vectors along a line that passes through the origin in $\mathbb{R}^2$ is a subspace. Do the same for $\mathbb{R}^3$. [Can you identify a spanning set?...]



2 Show that all vectors along a plane that passes through the origin form a subspace in $\mathbb{R}^3$. [Can you identify a spanning set?]

3 What do you think happens in $\mathbb{R}^4$? $\mathbb{R}^5$? $\mathbb{R}^n$?

If you did the last exercise, then you understand geometrically what almost all subspaces look like. There were a couple not mentioned ... because they are kind of silly:

EXAMPLE 2

just a set consisting of one vector, $\vec{0}$.

Show that $\{\vec{0}\}$ and $\mathbb{R}^n$ are both subspaces of $\mathbb{R}^n$.

First, let's show $\{\vec{0}\}$ is a subspace:

① clearly $\vec{0}$ is in $\{\vec{0}\}$.

② if $\vec{u}, \vec{v} \in \{\vec{0}\}$, then $\vec{u} = \vec{0}$, $\vec{v} = \vec{0}$ and $\vec{u} + \vec{v} = \vec{0}$, so condition ② is satisfied.

③ if $\vec{u} \in \{\vec{0}\}$, then $\vec{u} = \vec{0}$ and $r\vec{0} = \vec{0}$ for any scalar r, so condition ③ is satisfied.

$\Rightarrow \{\vec{0}\}$ is a subspace of $\mathbb{R}^n$. (we could also notice $\{\vec{0}\} = \text{span}\{\vec{0}\}$.)

Next, we'll show $\mathbb{R}^n$ is a subspace:

- First, $\vec{0}$ is in $\mathbb{R}^n$.. ✓
- Next, notice any vector in $\mathbb{R}^n$ is in the span of $\vec{e}_1, .., \vec{e}_n$, where $\vec{e}_j$ are as we defined them in lecture #8 (3.3).

$$\left(x_1 \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} + \cdots + x_n \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \right).$$

$\Rightarrow$ Since we see $\mathbb{R}^n = \text{span}\{\vec{e}_1, .., \vec{e}_n\}$, we are done! $\mathbb{R}^n$ is a subspace.

DEF We call $\{\vec{0}\}$ and $\mathbb{R}^n$ $\boxed{\text{trivial subspaces}}$, not because they are "easy", but because they are a little silly. They don't tell us much!

EXAMPLE 3

Let S be the subset of $\mathbb{R}^4$ consisting of vectors of the form $\begin{bmatrix} 2s-5t \\ 3s \\ 5r+3s-t \\ 3r+t \end{bmatrix}$, where r, s, t are any real numbers.

Is S a subspace?

Check this out:

$$\begin{bmatrix} 2s-5t \\ 3s \\ 5r+3s-t \\ 3r+t \end{bmatrix} = r\begin{bmatrix} 0 \\ 0 \\ 5 \\ 3 \end{bmatrix} + s\begin{bmatrix} 2 \\ 3 \\ 3 \\ 0 \end{bmatrix} + t\begin{bmatrix} -5 \\ 0 \\ -1 \\ 1 \end{bmatrix}$$

$$= \text{span}\left\{ \begin{bmatrix} 0 \\ 0 \\ 5 \\ 3 \end{bmatrix}, \begin{bmatrix} 2 \\ 3 \\ 3 \\ 0 \end{bmatrix}, \begin{bmatrix} -5 \\ 0 \\ -1 \\ 1 \end{bmatrix} \right\}$$

$\Rightarrow S$ is a subspace.

EXERCISE Let S be the subset of $\mathbb{R}^4$ consisting of vectors of the form

$$\vec{v} = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{bmatrix}$$

where $v_1 + v_2 + v_3 + v_4 = 0$. Is S a subspace?

(Hint: Check conditions directly. If you get stuck, check out example 7 in 4.1 in the textbook.)

EXERCISE

Show the set S that consists of all vectors of the form $\begin{bmatrix} 1+t \\ t \end{bmatrix}$, where t is any real number is not a subspace.

[Hint: Think about it geometrically first. Does that tell you which condition breaks?]

Now, notice that in the EXAMPLE 5, what we saw kind of looked like a solution to a homogeneous system ... i.e. the parameter part... Turns out that all such solutions are subspaces!

THM If A is an $n \times m$ matrix, then the set of solutions to the homogeneous linear system $A\vec{x} = \vec{0}$ forms a subspace of $\mathbb{R}^m$.

Notice, if the only solution is the trivial solution, then $\vec{x} = \vec{0}$, and $\{\vec{0}\}$ is a subspace. Otherwise, there are parameters, and we see the solution as a span of vectors:

EXAMPLE 4

Let $A = \begin{bmatrix} 2 & 1 & -1 \\ 1 & -1 & 5 \\ -2 & 2 & -10 \end{bmatrix}$. Solve $A\vec{x} = \vec{0}$.

By using Gauss-Jordan Elimination, we can find

$$\left[\begin{array}{ccc|c} 2 & 1 & -1 & 0 \\ 1 & -1 & 5 & 0 \\ -2 & 2 & -10 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 0 & \frac{4}{3} & 0 \\ 0 & 1 & -\frac{11}{3} & 0 \\ 0 & 0 & 0 & 0 \end{array}\right], \quad \text{so } \vec{x} = s\begin{bmatrix} 4/3 \\ -11/3 \\ 1 \end{bmatrix}.$$

i.e. solution set = $\text{span}\left\{ \begin{bmatrix} 4/3 \\ -11/3 \\ 1 \end{bmatrix} \right\}$.

<u>EXERCISE</u> Look at the exercise 2 pages back. Notice that the condition $v_1 + v_2 + v_3 = 0$ is a homogeneous linear equation:

$$\begin{bmatrix} 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \vec{0} \leftarrow [0]$$

Now, use the <u>THM</u> to conclude S is a subspace.

It turns out that this <u>solution set</u> for any homogeneous equation is important. So important, we give it <u>two different names!</u> (why? Blegh... one is for a "matrix", the other for the equivalent condition for a linear transformation...)

<u>DEF</u> If A is an $n \times m$ matrix, then the set of solutions to $A\vec{x} = \vec{0}$ is called the $\boxed{\text{null space}}$ <u>of A</u> and is denoted $\boxed{\text{null}(A)}$.

Similarly,

<u>DEF</u> If T is a linear transformation, then the set of vectors $\vec{x}$ such that $T(\vec{x}) = \vec{0}$ is called the $\boxed{\text{kernel of } T}$ and is denoted $\boxed{\text{Ker}(T)}$

Notice, however, even though we know null(A) is a subspace... and we may suspect ker(T) is a subspace, we still need to show that. But it is really easy!

$\boxed{\text{THM}}$ <u>THM</u> Let $T: \mathbb{R}^m \to \mathbb{R}^n$ be a linear transformation. Then $\ker(T)$ is a subspace of the domain.

pf: Let A be the matrix such that $T(\vec{x}) = Ax$. Since $\text{null}(A)$ is a subspace, $\ker(T)$ is a subspace.

Next, notice that the vectors $\vec{x}$ such that $T(\vec{x}) = \vec{0}$ live in the domain $\mathbb{R}^m$, so $\ker(T)$ is a subspace of $\mathbb{R}^m$, the domain! ☐

In fact, there is a little more... the range of a linear transformation (also called the image) is also a subspace!

<u>THM</u> Let $T: \mathbb{R}^m \to \mathbb{R}^n$ be a linear transformation. Then $\text{range}(T)$ is a subspace of the codomain.

pf: Let A be the matrix such that $T(\vec{x}) = Ax$. Then

$$A = [\vec{a}_1 \ \cdots \ \vec{a}_m]_{n \times m}$$

for a set of vectors in $\mathbb{R}^n$.

Then

$$A\vec{x} = [\vec{a}_1 \ \cdots \ \vec{a}_m] \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_m \end{bmatrix}$$

$$= x_1\vec{a}_1 + x_2\vec{a}_2 + \cdots + x_m\vec{a}_m,$$

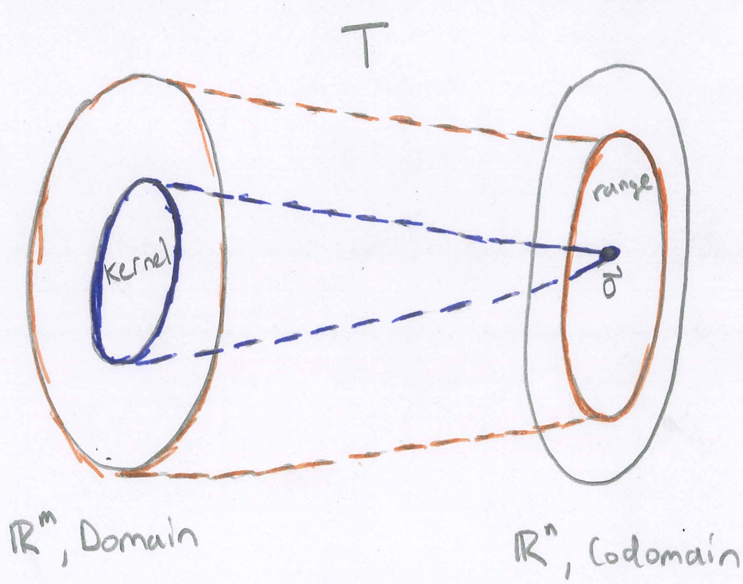
i.e. any vector in the Range of T is a linear combination of $\{\vec{a}_1, \ldots, \vec{a}_m\}$, which means

$$\text{range}(T) = \text{span}\{\vec{a}_1, \ldots, \vec{a}_m\},$$

so we see range of T is a subspace.

Next recall that the range is in the codomain. You can also see this by noticing you are taking a linear combination of a set of vectors in $\mathbb{R}^n$, the codomain! ☒

There is a nice picture that goes along with these two theorems:



Next, recall a fact we discovered in Lecture #6 (3.1):

If T is a linear transformation, then T is one-to-one if and only if $T(\vec{x}) = \vec{0}$ only has the trivial solution. i.e., the only solution to $T(\vec{x}) = \vec{0}$ is $\vec{x} = \vec{0}$, which is the ker(T)! So, we have the following fact:

THM Let $T : \mathbb{R}^m \to \mathbb{R}^n$ be a linear transformation. Then T is one-to-one if and only if $\ker(T) = \{\vec{0}\}$.

And, now, we can add this to a special case where $T: \mathbb{R}^n \to \mathbb{R}^n$...

i.e. we can update our unifying theorem!

THM (Unifying Theorem, version 4)

Let $S = \{\vec{a}_1, \ldots, \vec{a}_n\}$ be a set of n vectors in $\mathbb{R}^n$, let $A = [\vec{a}_1 \cdots \vec{a}_n]_{n \times n}$, and let $T(\vec{x}) = A\vec{x}$ where $T: \mathbb{R}^n \to \mathbb{R}^n$. Then the following are equivalent:

(a) S spans $\mathbb{R}^n$

(b) S is linearly independent.

(c) $A\vec{x} = \vec{b}$ has a unique solution for all $\vec{b}$ in $\mathbb{R}^n$.

(d) T is onto.

(e) T is one-to-one.

(f) A is invertible.

$\boxed{\text{(g) } \ker(T) = \{0\}.}$

EXERCISE Let A be an invertible matrix. Is it possible that $\ker(T) \neq \{0\}$? If A is not invertible, can $\ker(T) = \{0\}$.

EXERCISE (T/F) ① Let A be a 3×6 matrix. Then $\ker(T)$ is a subspace of $\mathbb{R}^6$.

② Let $T : \mathbb{R}^3 \to \mathbb{R}^6$ be a linear transformation. Then $\ker(T)$ is a subspace of $\mathbb{R}^6$.