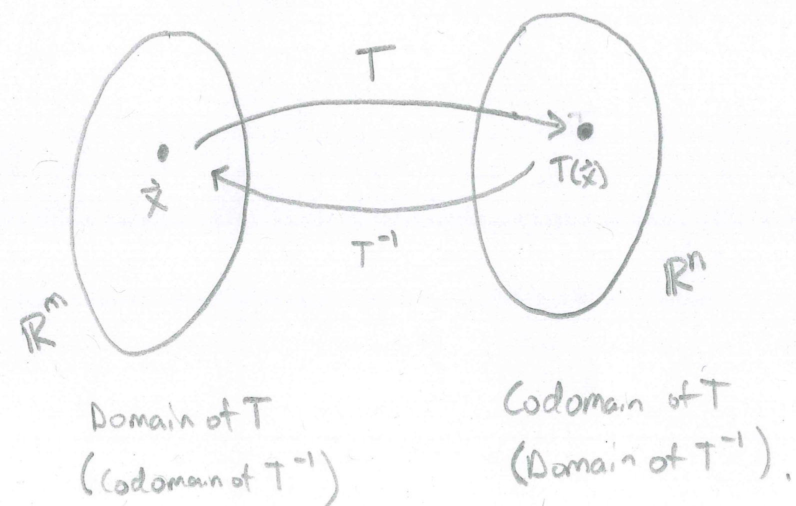
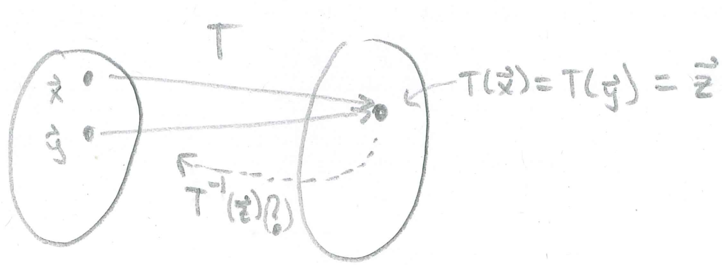
Lecture #8

3.3 Inverses

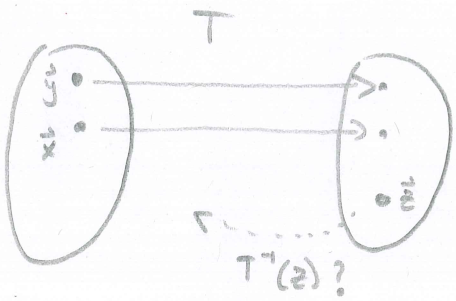
Our goal in this set of lecture notes is to understand when, given a linear transformation, we can go "backwards," and then learn to compute this "backward" transformation. In other words, we are looking for an inverse transformation. More precisely, given any transformation T, we understand where this transformation sends each vector $\vec{x}$ in the domain, so we want a transformation T^{-1} that sends $T(\vec{x})$ back to $\vec{x}$, for every $\vec{x}$.



We should notice two things immediately. First, if T is not one-to-one, then there cannot be an inverse transformation:



Similarly, if T is not onto, then there cannot be an inverse:



NOTE This is saying that if the codomain of T is not the range, then the codomain of T cannot be the domain of T^{-1}.

If you are wondering why you couldn't "restrict" the domain of T^{-1} to be the range of T when T is not onto, you could, but then you would want $T: \mathbb{R}^n \to \text{range}(T) \neq \mathbb{R}^m$, and that is not how we defined a linear transformation.

So, if we have a (linear!) transformation that is both one-to-one and onto, then we can define an inverse.

DEF A linear transformation $T: \mathbb{R}^m \to \mathbb{R}^n$ is $\boxed{\text{invertible}}$ if T is one-to-one and onto. When T is invertible, the $\boxed{\text{inverse function}}$ $T^{-1}: \mathbb{R}^n \to \mathbb{R}^m$ is defined by

$$T^{-1}(\vec{y}) = \vec{x} \quad \text{if and only if} \quad T(\vec{x}) = \vec{y}.$$

Notice, this means
$$\begin{cases} \text{①} \ T(T^{-1}(\vec{y})) = \vec{y} \\ \text{②} \ T^{-1}(T(\vec{x})) = \vec{x} \end{cases}$$

Recall that if $m<n$, $T:\mathbb{R}^m \to \mathbb{R}^n$ cannot be onto. If $m>n$, then T cannot be one-to-one. In other words..

THM Let $T:\mathbb{R}^m \to \mathbb{R}^n$ be a linear transformation. Then

(a) The only way T has an inverse is if $\boxed{m=n}$.

(b) If T is invertible, then T^{-1} is also a linear transformation.

To see (b), just assume T is a linear transformation, and use the fact that $T^{-1}(\vec{y})=\vec{x}$ if and only if $T(\vec{x})=\vec{y}$, and you can show that T^{-1} satisfies the definition of a linear transformation. See p. 131/132 in the book for the rigorous proof if you are interested.

For us, however, we are more interested in the fact that linear transformations are represented by matrices ... So we should be able to translate this abstract idea of an inverse into a concrete one: what does the inverse of a matrix look like? And how do we compute it? And if we can compute it ... how would we use it?

First, let's translate what we know about linear transformations into matrices. For a matrix A, to be invertible, it <u>must</u> be square $(m=n)$. Similarly, we need for $A^{-1}A\vec{x} = \vec{x}$ and $AA^{-1}\vec{y} = \vec{y}$.

Notice the following: $I_n : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ is given by

$$\left(I_n \quad \vec{x} \quad = \quad \vec{x} \right)$$

$$\begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \\ 0 & 0 & & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix},$$

so this means what we are really looking for is an A^{-1} such that $AA^{-1} = I_n$ and $A^{-1}A = I_n$.

<u>DEF</u> An $n \times n$ matrix A is $\boxed{\underline{\text{invertible}}}$ if there exists an $n \times n$ matrix B such that $AB = I_n$.

<u>THM</u> Suppose A is an invertible matrix with $AB = I_n$. Then $BA = I_n$, and the matrix B such that $AB = BA = I_n$ is unique.

So,

<u>DEF</u> If an $n \times n$ matrix A is invertible, we let $\boxed{\underline{A^{-1}}}$ denote the unique inverse of A, and we have

(*) $\boxed{A^{-1}A = AA^{-1} = I_n.}$ (*)

To see why the inverse is unique, check the proof on p. 133 in the text. The technique should remind you of a technique we used in an earlier lecture.

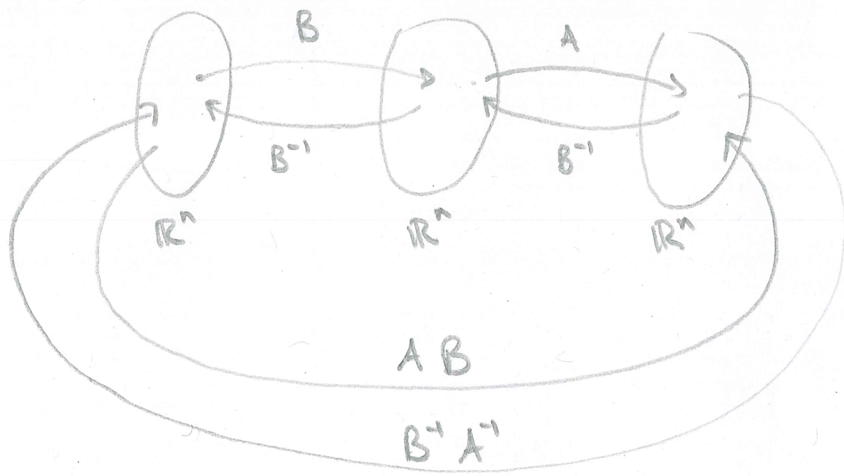
Let's identify a few abstract properties of inverse matrices, then we'll do some examples showing how to compute the inverse.

PROPERTIES

Let A and B be invertible $n \times n$ matrices and C and D be $n \times m$ matrices. Then

(a) A^{-1} is invertible, and $(A^{-1})^{-1} = A$.

(b) AB is invertible, and $(AB)^{-1} = B^{-1}A^{-1}$



c) If $AC = AD$, then $C = D$.

d) If $AC = 0_{nm}$, then $C = 0$

$\Leftarrow$ this "fixes" some of the problems we had with multiplication!

Now, we turn our attention to <u>finding</u> the inverse.

Here's the idea: We want to find A^{-1} where $AA^{-1} = I_n$.

Let $\vec{a_1^{-1}}$ denote the first column of A^{-1}, $\vec{a_2^{-1}}$ denote the second, and so on. Then, our goal is to solve a n systems of equations simultaneously:

$$\begin{cases} A\vec{a_1^{-1}} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \leftarrow \text{first column in } I_n \\[2em] A\vec{a_2^{-1}} = \begin{bmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \leftarrow \text{second column in } I_n \\ \quad\vdots \\ A\vec{a_n^{-1}} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix} \leftarrow n^{th} \text{ column in } I_n. \end{cases}$$

(<u>NOTE</u> If you are finding it difficult to see why this is true, return to the exercise on p. 12 of the Lecture #7 notes, then put the pieces together.)

Now, we are going to use the unifying theorem. A is a square matrix, and A is one-to-one (and onto), so by the unifying theorem, we see that each of the systems of equations above have one unique solution (which makes sense, since the inverse will be unique). This means that if we use Gauss-Jordan elimination, then the matrix A in reduced row echelon form will be:

$$A \sim \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & \vdots \\ \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & \cdots & 1 \end{bmatrix} = I_n,$$

i.e. for the first system

first colum of I_n — the unique solution!

$$\left[\begin{array}{c|c} A & \begin{matrix} 1 \\ 0 \\ \vdots \\ 0 \end{matrix} \end{array}\right] \sim \left[\begin{array}{cccc|c} 1 & 0 & \cdots & 0 & \\ 0 & 1 & & \vdots & \vec{a}_1^{-1} \\ \vdots & & \ddots & \vdots & \\ 0 & \cdots & \cdots & 1 & \end{array}\right].$$

That means that we can do Gauss-Jordan Elimination once on A to get all columns of $\vec{a}_i^{-1}$ simultaneously:

$$\left[\begin{array}{c|cccc} & 1 & 0 & \cdots & 0 \\ & 0 & 1 & \cdots & \vdots \\ A & 0 & 0 & \ddots & \vdots \\ & \vdots & \vdots & & \vdots \\ & 0 & 0 & \cdots & 1 \end{array}\right] \xrightarrow{\text{Gauss-Jordan}} \left[\begin{array}{cccc|c} 1 & 0 & \cdots & 0 & \\ 0 & 1 & \cdots & 0 & \\ \vdots & \vdots & \ddots & \vdots & \overbrace{\vec{a}_1^{-1}\ \vec{a}_2^{-1} \cdots \vec{a}_n^{-1}}^{A^{-1}} \\ 0 & 0 & \cdots & 1 & \end{array}\right]$$

So, to find the inverse, all we do is setup the augmented matrix $[A \mid I_n]$, run Gauss-Jordan elimination, and we get $[I_n \mid A^{-1}]$, where $[A \mid I_n] \sim [I_n \mid A^{-1}]$.

At this point, it is probably worth defining $\vec{e}_1, \vec{e}_2, \ldots, \vec{e}_n$ the way the book does it (these become very important in the next chapter):

DEF ($\vec{e}_j$)

We often denote the j^{th}-column of I_n as $\boxed{\vec{e}_j}$.

For example, $\vec{e}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$, $\vec{e}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$, $\ldots$, $\vec{e}_n = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}$.

This means that we could rewrite the systems of equations from 2 pages back as:

$$\begin{cases} A\vec{a}_1^{-1} = \vec{e}_1 \\ A\vec{a}_2^{-1} = \vec{e}_2 \\ \vdots \\ A\vec{a}_n^{-1} = \vec{e}_n \end{cases}.$$

Now, finally, an example!

EXAMPLE 1

Find the inverse of $A = \begin{bmatrix} 1 & 3 \\ 2 & 5 \end{bmatrix}$.

Follow our procedure! $\left[\begin{array}{cc|cc} 1 & 3 & 1 & 0 \\ 2 & 5 & 0 & 1 \end{array}\right] \overset{-2R_1 + R_2 \to R_2}{\sim} \left[\begin{array}{cc|cc} 1 & 3 & 1 & 0 \\ 0 & -1 & -2 & 1 \end{array}\right]$

$$\overset{-1R_2 \to R_2}{\sim} \left[\begin{array}{cc|cc} 1 & 3 & 1 & 0 \\ 0 & 1 & 2 & -1 \end{array}\right]$$

$$\xrightarrow{-3R_2+R_1 \to R_1} \sim \left[\begin{array}{cc|cc} 1 & 0 & -5 & 3 \\ 0 & 1 & 2 & -1 \end{array}\right]$$

So $A^{-1} = \begin{bmatrix} -5 & 3 \\ 2 & -1 \end{bmatrix}$.

<u>EXERCISE</u> Check that $AA^{-1} = A^{-1}A = I_2$.

On your homework, you will do this for larger matrices!

Next, notice that we can use an inverse matrix to solve for some $\vec{x}$. Given $\vec{b}$, and an invertible matrix A, if we know A^{-1}, then

$$A\vec{x} = \vec{b}$$

multiply both sides by A^{-1} on the left.

$$A^{-1}A\vec{x} = A^{-1}\vec{b}$$

$$I_n\vec{x} = A^{-1}\vec{b}$$

$$\vec{x} = A^{-1}\vec{b}. \qquad (!)$$

<u>EXAMPLE 2</u>

Find the unique solution to the linear system

$$x_1 + 3x_2 = 4$$
$$2x_1 + 5x_2 = -3.$$

Set-up an augmented matrix

$$\Rightarrow \left[\begin{array}{cc|c} 1 & 3 & 4 \\ 2 & 5 & -3 \end{array}\right]$$

but before you solve, rewrite this as a matrix equation $A\vec{x} = \vec{b}$:

$$A\vec{x} = \begin{bmatrix} 4 \\ -3 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 3 \\ 2 & 5 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 4 \\ -3 \end{bmatrix}$$

Notice, we know the inverse matrix for A. (From the last example.) Then

$$A^{-1}A\vec{x} = A^{-1}\begin{bmatrix} 4 \\ -3 \end{bmatrix}$$

$$\vec{x} = A^{-1}\begin{bmatrix} 4 \\ -3 \end{bmatrix}, \quad A^{-1} = \begin{bmatrix} -5 & 3 \\ 2 & -1 \end{bmatrix}$$

$$\vec{x} = \begin{bmatrix} -5 & 3 \\ 2 & -1 \end{bmatrix}\begin{bmatrix} 4 \\ -3 \end{bmatrix}$$

$$\vec{x} = \begin{bmatrix} -29 \\ 11 \end{bmatrix}. \Rightarrow \begin{cases} x_1 = -29 \\ x_2 = 11 \end{cases} \quad \boxtimes$$

Earlier, when constructing our procedure for computing the inverse, we noticed that since A is square, and one-to-one, then by the unifying theorem, it is <u>onto</u>, the <u>columns span $\mathbb{R}^n$</u>, the columns are linearly independent, and $A\vec{x}=\vec{b}$ has one unique solution for every $\vec{b}$. (this last one is the fact we used).

This actually tells us a lot about invertible matrices:

THM Let A be an $n \times n$ matrix. Then the following are equivalent

(a) A is invertible

(b) $A\vec{x}=\vec{b}$ has a unique solution for all $\vec{b}$, given by $\vec{x}=A^{-1}\vec{b}$.

(c) $A\vec{x}=\vec{0}$ has only the trivial solution.

(Invertible linear transformations always map $\vec{0}$ to $\vec{0}$!)

We can also update our unifying Theorem again:

THM (Unifying Theorem, ver. 3)
Let $S=\{\vec{a}_1, \ldots, \vec{a}_n\}$ be a set of n vectors in $\mathbb{R}^n$, let $A=[\vec{a}_1 \cdots \vec{a}_n]$, and let $T:\mathbb{R}^n \to \mathbb{R}^n$ be given by $T(\vec{x})=A\vec{x}$. Then the following are equivalent:

(a) S spans $\mathbb{R}^n$

(b) S is linearly independent

(c) $A\vec{x}=\vec{b}$ has one unique solution for all $\vec{b}$ in $\mathbb{R}^n$.

(d) T is onto

(e) T is one-to-one

(f) A is invertible. (*) (*)

One last note, you could use our procedure (for computing the inverse of a matrix) on general matrices to generate formulas. This is actually quite useful for 2×2 matrices, so we give a formula below:

!!!

(*) **Formula for the inverse of a 2×2 Matrix** (*)

If $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, then $A^{-1} = \dfrac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$.

(*) **EXERCISE** Use Gauss-Jordan to prove this! i.e. Use our procedure:

$$[A \mid I_n] \sim [I_n \mid A^{-1}]$$

$$\begin{bmatrix} a & b & 1 & 0 \\ c & d & 0 & 1 \end{bmatrix}.$$

EXERCISE Notice that we must divide by the quantity $ad-bc$. Does this give us a condition for when A is invertible? (i.e. Can $ad-bc=0$?) What do you think this means in terms of the span of the column vectors of A? Whether or not they are linearly independent? It turns out that for a 2×2 matrix, this quantity $ad-bc$ is very important! It will come back in Chapter 5.

Finally, let's introduce a few terms that will be used moving forwards.

DEF (singular, nonsingular)

A square matrix A that is invertible is called $\boxed{\text{nonsingular}}$.
If A does not have an inverse, it is called $\boxed{\text{singular.}}$

EXERCISE

Work through the conceptual problem for Chapter 3. You can find these on the course website under the "Other Materials" heading. Additionally, watch the chapter 3 videos. Links to these are also on the course website under the "other materials" heading.