Lecture #5

2.3 Linear Independence

DEF Let $\{u_1, u_2, \dots, u_m\}$ be a set of vectors in $\mathbb{R}^n$. If the only solution to the vector equation

$$x_1 \vec{u}_1 + x_2 \vec{u}_2 + \cdots + x_m \vec{u}_m = \vec{0}$$

is $x_1 = x_2 = \cdots = x_m = 0$ (the trivial solution) then the set $\{\vec{u}_1, \vec{u}_2, \dots, \vec{u}_m\}$ is **linearly independent**. If there are non-trivial solutions, then the set is **linearly dependent**.

EXAMPLE 1

Let $\vec{u}_1 = \begin{bmatrix} -1 \\ 5 \\ 4 \end{bmatrix}$, $\vec{u}_2 = \begin{bmatrix} 4 \\ -4 \\ 0 \end{bmatrix}$, and $\vec{u}_3 = \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}$. Is the set linearly dependent?

Consider the equation:

$$x_1 \begin{bmatrix} -1 \\ 5 \\ 4 \end{bmatrix} + x_2 \begin{bmatrix} 4 \\ -4 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

We want to know if this has one solution $(x_1=0, x_2=0, x_3=0)$ or infinitely many. Turn it into a matrix and compute.

$$\left[\begin{array}{ccc|c} -1 & 4 & 1 & 0 \\ 5 & -4 & -1 & 0 \\ 4 & 0 & 2 & 0 \end{array}\right] \overset{\substack{5R_1+R_2 \to R_2 \\ 4R_1+R_3 \to R_3}}{\sim} \left[\begin{array}{ccc|c} -1 & 4 & 1 & 0 \\ 0 & 16 & 4 & 0 \\ 0 & 16 & 6 & 0 \end{array}\right]$$

$$\overset{R_3 + -1 \cdot R_2 \to R_2}{\sim} \left[\begin{array}{ccc|c} -1 & 4 & 1 & 0 \\ 0 & 16 & 4 & 0 \\ 0 & 0 & 2 & 0 \end{array}\right]$$

Then, you could use Gauss-Jordan Elimination, but we will go ahead and turn it back into a system and use back substitution:

$$\begin{cases} -x_1 + 4x_2 + x_3 = 0 \\ 16x_2 + 4x_3 = 0 \\ 2x_3 = 0 \end{cases}$$

Since $2x_3 = 0$, we see $x_3 = 0$. Then

$$16x_2 + 4(0) = 0$$
$$16x_2 = 0 \implies x_2 = 0.$$

And finally:

$$-x_1 + 4(0) + 0 = 0 \implies x_1 = 0.$$

So we see there is one solution. Thus, the set is linearly independent.

EXAMPLE 2

Is the set $\begin{bmatrix}1\\0\\0\end{bmatrix}, \begin{bmatrix}0\\1\\0\end{bmatrix}, \begin{bmatrix}0\\0\\1\end{bmatrix}, \begin{bmatrix}5\\0\\3\end{bmatrix}$ a linearly independent set?

Look for solutions to:

$$x_1\begin{bmatrix}1\\0\\0\end{bmatrix} + x_2\begin{bmatrix}0\\1\\0\end{bmatrix} + x_3\begin{bmatrix}0\\0\\1\end{bmatrix} + x_4\begin{bmatrix}5\\0\\3\end{bmatrix} = \begin{bmatrix}0\\0\\0\end{bmatrix}$$

Turn it into a matrix:

$$\left[\begin{array}{cccc|c} 1 & 0 & 0 & 5 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 3 & 0 \end{array}\right]$$

It's already in reduced row echelon form! We are basically looking at the solution. Let's put it in equation form:

$$\begin{cases} x_1 \quad + 5x_4 = 0 \\ x_2 = 0 \\ x_3 + 3x_4 = 0 \end{cases}$$

x_4 is a free variable, so let $x_4 = s$ (a parameter). Then

$$x_3 = -3s$$
$$x_2 = 0$$
$$x_1 = -5s,$$

And we see there are an infinite # of solutions! So not linearly independent no.

In the last example, notice that we had $\boxed{\underline{\underline{4}}}$ vectors in $\mathbb{R}^{\boxed{\underline{\underline{3}}}}$, and $\boxed{4>3}$. This means that the corresponding system of equations will have an x_1, x_2, x_3 and x_4. Since we are in three "dimensions", there are only three equations. (Verify this by looking at the example!) This means that we will <u>always</u> have a free variable, so the solution will <u>always</u> have a parameter, so there will <u>always</u> be an infinite number of solutions. In other words, any set of 4 vectors in $\mathbb{R}^3$ will <u>always</u> be linearly dependent. This works in higher dimensions, where the crux of our argument here is that the number of vectors in the set is greater than the "dimension".

<u>THM</u> Suppose that $\{\vec{u}_1, \vec{u}_2, \ldots, \vec{u}_m\}$ is a set of vectors in $\mathbb{R}^n$. If $n < m$, then the set is linearly dependent. (i.e <u>Not</u> linearly independent.)

Next, let's ask a question ... what if our set of vectors $\{\vec{0}, \vec{u}_2, \ldots, \vec{u}_m\}$ contained the zero vector, $\vec{0}$?

Consider the equation:

$$x_1\vec{0} + x_2\vec{u}_2 + \cdots + x_m\vec{u}_m = \vec{0}.$$

There is at least <u>one</u> solution, $x_1 = x_2 = \cdots = x_m = 0$. Are there more? <u>Yes!</u> We could let x_1 be <u>anything</u>, and let $x_2 = x_3 = \cdots = x_m = 0$, so we will have an infinite number of solutions. Thus, $\{\vec{0}, \vec{u}_2, \ldots, \vec{u}_m\}$ is a linearly dependent set.

<u>THM</u> Suppose that $\{\vec{0}, \vec{u}_2, \vec{u}_3, \ldots, \vec{u}_m\}$ is a set of vectors in $\mathbb{R}^n$. Then the set is linearly dependent.

Okay, so given the last two theorems, it may feel like our notion of linear independence and span should be somehow connected. They are certainly not the same, but they are related to each other in the following way:

THM Let $\{\vec{u}_1, \vec{u}_2, \ldots, \vec{u}_m\}$ be a set of vectors in $\mathbb{R}^n$. Then this set is linearly dependent if and only if one of the vectors in the set is in the span of the other vectors.

This is a theorem we are going to prove...

pf: We will first show that if the set is linearly dependent, then one of the vectors is in the span of the other vectors. (We will also have to show that if one of the vectors is in the span of the other vectors, the set is linearly dependent. This is what allows us to say <u>if and only if</u>.)

Assume that the set is linearly dependent. Then

$$x_1\vec{u}_1 + x_2\vec{u}_2 + \cdots + x_m\vec{u}_m = \vec{0}$$

has an infinite number of solutions. Pick any non-trivial solution, then at least one of $x_1, x_2, \ldots, x_m$ must be non-zero. Let's assume it is x_1 (otherwise, reorder your vectors so it is x_1). Then, solve for $\vec{u}_1$:

$$x_1\vec{u}_1 + x_2\vec{u}_2 + \cdots + x_m\vec{u}_m = \vec{0}$$

$$x_1\vec{u}_1 = -x_2\vec{u}_2 + -x_3\vec{u}_3 + \cdots + -x_m\vec{u}_m$$

since $x_1 \neq 0$, we can divide!

$$\vec{u}_1 = \frac{-x_2}{x_1}\vec{u}_2 + \frac{-x_3}{x_1}\vec{u}_3 + \cdots + \frac{-x_m}{x_1}\vec{u}_m.$$

Thus, $\vec{u}_1$ is in the span of $\{\vec{u}_2, \vec{u}_3, .., \vec{u}_m\}$.

Now, let's go the other way (to show the <u>converse</u>). Assume one of the vectors in the set is in the span of the others. Again, without loss of generality, we can assume $\vec{u}_1$ is in the span of the others (otherwise, reorder the vectors). Then

$$\vec{u}_1 = c_2\vec{u}_2 + c_3\vec{u}_3 + \cdots + c_m\vec{u}_m$$

for some constants $c_2, c_3, \ldots, c_m$. Notice, if $\vec{u}_1$ is the zero vector $\vec{0}$, then an earlier theorem tells us the set is linearly dependent (we are done!), so we can assume $\vec{u}_1$ is <u>not</u> the zero vector. This means at least one of $c_2, c_3, .., c_m$ must be non-zero.

Now, move all terms in the equation above to one side:

$$\vec{u}_1 - c_2\vec{u}_2 - c_3\vec{u}_3 - \cdots - c_m\vec{u}_m = \vec{0},$$

and we see that we have a <u>non-trivial</u> solution. In other words, there are infinitely many solutions to the equation

$$x_1\vec{u}_1 + x_2\vec{u}_2 + \cdots + x_m\vec{u}_m = \vec{0}.$$

This proves the "if and only if" statement in the theorem.

$\blacksquare$

<u>EXERCISE</u> Work example 3 on page 82 of the text.

Now, let's expand a little on this relationship between span and linear independence.

<u>THM</u> Let $\vec{u}_1, \ldots, \vec{u}_m$ be vectors in $\mathbb{R}^n$. Suppose

$$A = [\vec{u}_1 \; \vec{u}_2 \; \ldots \; \vec{u}_m] \sim B$$

where B is in echelon form. Then

(a) <u>$\text{span}\{\vec{u}_1, \ldots, \vec{u}_m\} = \mathbb{R}^n$</u> exactly when B has a pivot position in every row.

(b) $\{\vec{u}_1, \ldots, \vec{u}_m\}$ is <u>linearly independent</u> exactly when B has a pivot position in every column.

EXERCISE Recall from 2.2 that we already knew part a) in this theorem. Complete the following sentence, and then use this to justify part b).

- If a matrix B in echelon form has a pivot position (think leading term) in every column, then the associated homogeneous system of equations is in ________ form.

[Hint: not echelon ... better ...]

Next, notice that we can re-phrase the definition of linear independence as follows:

THM Let $A = [\vec{a}_1 \ \vec{a}_2 \ \cdots \ \vec{a}_m]$ and $\vec{x} = (x_1, \ldots, x_m)$. The set $\{\vec{a}_1, \ldots, \vec{a}_m\}$ is linearly independent if and only if the homogeneous linear system

$$A\vec{x} = \vec{0}$$

has only the trivial solution.

EXAMPLE 3

Find the general solution to the following $\boxed{\text{non-homogeneous}}$ (just think "not homogeneous") linear system, then find the solution to the homogeneous system:

$\boxed{\text{non-homogeneous:}}$

$$\begin{cases} 2x_1 - 6x_2 - x_3 + 8x_4 = 7 \\ x_1 - 3x_2 - x_3 + 6x_4 = 6 \\ -x_1 + 3x_2 - x_3 + 2x_4 = 4 \end{cases}$$

$\boxed{\text{homogeneous:}}$

$$\begin{cases} 2x_1 - 6x_2 + x_3 + 8x_4 = 0 \\ x_1 - 3x_2 - x_3 + 6x_4 = 0 \\ -x_1 + 3x_2 - x_3 + 2x_4 = 0 \end{cases}$$

Let's solve the general system first. By doing Gauss-Jordan elimination, we can find

(check!)

$$\left[\begin{array}{cccc|c} 2 & -6 & -1 & 8 & 7 \\ 1 & -3 & -1 & 6 & 6 \\ -1 & 3 & -1 & 2 & 4 \end{array}\right] \sim \left[\begin{array}{cccc|c} 1 & -3 & 0 & 2 & 1 \\ 0 & 0 & 1 & -4 & -5 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

which tells us:

$x_4 = s_1$ (parameter)

$x_3 = -5 + 4s_1$

$x_2 = s_2$ (parameter)

$x_1 = 1 - 2s_1 + 3s_2$,

and in vector form:

$$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ -5 \\ 0 \end{bmatrix} + s_1 \begin{bmatrix} -2 \\ 0 \\ 4 \\ 1 \end{bmatrix} + s_2 \begin{bmatrix} 3 \\ 1 \\ 0 \\ 0 \end{bmatrix}.$$

you'll notice the row operations are the same as above...

Next, we can solve the homogeneous system

(check!)

$$\left[\begin{array}{cccc|c} 2 & -6 & -1 & 8 & 0 \\ 1 & -3 & -1 & 6 & 0 \\ -1 & 3 & -1 & 2 & 4 \end{array}\right] \sim \left[\begin{array}{cccc|c} 1 & -3 & 0 & 2 & 0 \\ 0 & 0 & 1 & -4 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

which tells us:

$$\begin{cases} x_4 = s_1 \\ x_3 = 4s_1 \\ x_2 = s_2 \\ x_1 = -2s_1 + 3s_2 \end{cases}$$

So in vector form, we have

$$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = s_1 \begin{bmatrix} -2 \\ 0 \\ 4 \\ 1 \end{bmatrix} + s_2 \begin{bmatrix} 3 \\ 1 \\ 0 \\ 0 \end{bmatrix}.$$

Notice that the solution to the homogeneous system is the same as the solution to the general system, except that the constants are gone. This works in general!

It comes from the following property:

<u>THM</u> Suppose that $A = [\vec{a}_1 \ \vec{a}_2 \ \dots \ \vec{a}_m]$ and let $\vec{x} = (x_1, x_2, \dots, x_m)$ and $\vec{y} = (y_1, y_2, \dots, y_m)$. Then

a) $A(\vec{x} + \vec{y}) = A\vec{x} + A\vec{y}$, and

b) $A(\vec{x} - \vec{y}) = A\vec{x} - A\vec{y}$.

EXERCISE Show that the theorem is true using our definition of $A\vec{x}$ from 2.2.

Now notice that any solution to a general system looks something like

$$\vec{x} = \underbrace{\begin{bmatrix} 1 \\ 0 \\ -5 \\ 0 \end{bmatrix}}_{\boxed{\text{particular solution}}} + \underbrace{s_1 \begin{bmatrix} -2 \\ 0 \\ 4 \\ 1 \end{bmatrix} + s_2 \begin{bmatrix} 3 \\ 1 \\ 0 \\ 0 \end{bmatrix}}_{\boxed{\text{homogeneous solution}}}$$

If we pick our parameters to be 0 ($s_1 = s_2 = 0$), this is the solution, i.e. a particular solution.

this is the solution to the homogeneous system.

Let $\vec{x}_p$ denote the particular solution and $\vec{x}_h$ denote the homogeneous solution. Then any solution has this form:

$$\boxed{\vec{x} = \vec{x}_p + \vec{x}_h.}$$

$$\boxed{\text{Notice: } \quad A\vec{x} = A(\vec{x}_p + \vec{x}_h) = A\vec{x}_p + \cancelto{\vec{0}}{A\vec{x}_h} = A\vec{x}_p.} \quad (*)$$

Let's record this information in a theorem!

THM Let $\vec{x}_p$ be a particular solution to

$$A\vec{x} = \vec{b}.$$

Then all solutions $\vec{x}_g$ to the equation above have the form $\vec{x}_g = \vec{x}_p + \vec{x}_h$, where $\vec{x}_h$ is a solution to the homogeneous system $A\vec{x} = \vec{0}$.

EXERCISE Read the proof of this on page 85. You will need to use the property we noted at the bottom of the page prior to this one.

Now, let's put some of this together. If a set of vectors is linearly independent, then the only homogeneous solution is $\vec{0}$. Since the homogeneous part of the solution contains all of the parameters, if the only homogeneous solution is $\vec{0}$, then solving a non-homogeneous equation using the same vectors can have <u>at</u> <u>most</u> one solution.

THM Let $\vec{a}_1, \vec{a}_2, \ldots, \vec{a}_m$ and $\vec{b}$ be vectors in $\mathbb{R}^n$. Then the following statements are equivalent.

a) The set $\{\vec{a}_1, \vec{a}_2, \ldots, \vec{a}_m\}$ is linearly independent.

b) The vector equation $x_1 \vec{a}_1 + x_2 \vec{a}_2 + \cdots + x_m \vec{a}_m = \vec{b}$ has at most one solution for every $\vec{b}$.

c) The linear system corresponding to $[\vec{a}_1\ \vec{a}_2 \cdots \vec{a}_m \,|\, \vec{b}]$ has at most one solution for every $\vec{b}$.

d) The equation $A\vec{x} = \vec{b}$, with $A = [\vec{a}_1\ \vec{a}_2 \cdots \vec{a}_m]$ and $\vec{x} = (x_1, x_2, \ldots, x_m)$ has at most one solution for every $\vec{b}$.

EXERCISE Compare this to the last theorem in Lecture #3 (2.1). What if a set vectors is both linearly independent and spans $\mathbb{R}^n$?

THM (Unifying Theorem, Version 1)
Let $S = \{\vec{a}_1, \vec{a}_2, \ldots, \vec{a}_n\}$ be a set of vectors in $\mathbb{R}^n$, and let $A = [\vec{a}_1\ \vec{a}_2 \cdots \vec{a}_n]$. Then the following are equivalent:

a) S spans $\mathbb{R}^n$

b) S is linearly independent

c) $A\vec{x} = \vec{b}$ has a unique solution for all $\vec{b}$ in $\mathbb{R}^n$.

(pf: Did you do the Exercise?)

EXERCISE

Determine if one of the vectors is in the span of the other vectors.

(a) $\vec{u} = \begin{bmatrix} 4 \\ -1 \\ 3 \end{bmatrix}$, $\vec{v} = \begin{bmatrix} 3 \\ 5 \\ -2 \end{bmatrix}$, $\vec{w} = \begin{bmatrix} -5 \\ 7 \\ -7 \end{bmatrix}$

(b) $\vec{u} = \begin{bmatrix} 1 \\ 7 \\ 8 \\ 4 \end{bmatrix}$, $\vec{v} = \begin{bmatrix} -1 \\ 3 \\ 5 \\ 2 \end{bmatrix}$, $\vec{w} = \begin{bmatrix} 3 \\ 1 \\ -2 \\ 0 \end{bmatrix}$

[HINT: Check to see if the vectors are linearly independent, then appeal to a theorem that relates span to linear independence.]

EXERCISE

Determine if $A\vec{x} = \vec{b}$ has a unique solution for every $\vec{b}$ in $\mathbb{R}^4$, where

$$A = \begin{bmatrix} 2 & 5 & -3 & 6 \\ -1 & 0 & 1 & -1 \\ 5 & 2 & -3 & 9 \\ 3 & -4 & 6 & 8 \end{bmatrix}, \quad \vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$$

[Hint: It does not. Can you give an example of one that does?]

EXERCISE

Review the conceptual problems for Chapter 2 posted on the course website under "Other Materials."