2.2 Span

DEF Let $\{\vec{u}_1, \vec{u}_2, \dots, \vec{u}_m\}$ be a set of vectors in $\mathbb{R}^n$. The $\boxed{\text{span}}$ of this set is denoted $\boxed{\text{span}\{\vec{u}_1, \vec{u}_2, \dots, \vec{u}_m\}}$ and is defined as the set of all linear combinations

$$x_1\vec{u}_1 + x_2\vec{u}_2 + \cdots + x_m\vec{u}_m$$

where $x_1, x_2, \dots, x_m$ can be any real number.

At the end of Lecture #3, we had a short discussion about when you can find a linear combination of vectors in $\mathbb{R}^2$ that can get you to any vector $\begin{bmatrix} a \\ b \end{bmatrix}$, and when you can't. We were really talking about the span of the vectors. You should have noticed that if the two vectors are not parallel (or anti-parallel), the span is all of $\mathbb{R}^2$. If the vectors are parallel, or anti-parallel, it is <u>not</u> all of $\mathbb{R}^2$, the span is a line. In other words, in that case, there are vectors in $\mathbb{R}^2$ that are <u>not</u> in the span of the two parallel vectors.

What if we consider vectors in $\mathbb{R}^3$?

EXAMPLE 1

Is $\vec{v} = \begin{bmatrix} -10 \\ -8 \\ 14 \end{bmatrix}$ in the span of $\vec{u}_1 = \begin{bmatrix} -1 \\ 4 \\ -3 \end{bmatrix}$ and $\vec{u}_2 = \begin{bmatrix} 2 \\ 8 \\ -7 \end{bmatrix}$?

Assume it is, i.e. that there is some x_1 and x_2 such that

$$x_1 \begin{bmatrix} -1 \\ 4 \\ -3 \end{bmatrix} + x_2 \begin{bmatrix} 2 \\ 8 \\ -7 \end{bmatrix} = \begin{bmatrix} -10 \\ -8 \\ 14 \end{bmatrix},$$

then try to solve:

convert to augmented matrix

$$\left[\begin{array}{cc|c} -1 & 2 & -10 \\ 4 & 8 & -8 \\ -3 & -7 & 14 \end{array}\right]$$

$4 \cdot R_1 + R_2 \to R_2$

$-3 \cdot R_1 + R_3 \to R_3$

$$\sim \left[\begin{array}{cc|c} -1 & 2 & -10 \\ 0 & 16 & -48 \\ 0 & -13 & 44 \end{array}\right]$$

$\frac{13}{16} R_2 + R_3 \to R_3$

$$\sim \left[\begin{array}{cc|c} -1 & 2 & -10 \\ 0 & 16 & -48 \\ 0 & 0 & -5 \end{array}\right]$$

So we see there is <u>no</u> solution. In other words,

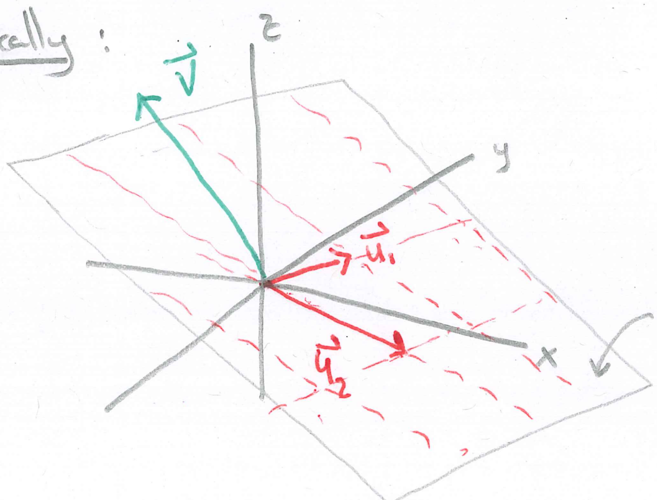
$\vec{v} = \begin{bmatrix} -10 \\ -8 \\ 14 \end{bmatrix}$ is <u>not</u> in the span of $\vec{u}_1$ and $\vec{u}_2$.

In different notation, we can write

$$\vec{v} \notin \text{span}\{\vec{u}_1, \vec{u}_2\}.$$

this just means "not in".

<u>Geometrically</u>:



plane containing $\vec{u}_1$ and $\vec{u}_2$.

linear combinations of $\vec{u}_1$ and $\vec{u}_2$ can take us anywhere in the plane, but not off of it! $\vec{v}$ is not in the plane, so we cannot find a linear combination of $\vec{u}_1$ and $\vec{u}_2$ to "get to $\vec{v}$."

EXAMPLE 2

Show that the span of $\vec{u}_1 = \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}$, $\vec{u}_2 = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$, and $\vec{u}_3 = \begin{bmatrix} 9 \\ 1 \\ -1 \end{bmatrix}$

is all of $\mathbb{R}^3$.

To do this, let $\vec{v} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$ be any arbitrary vector

in $\mathbb{R}^3$ and show that we can always find real numbers

x_1, x_2, and x_3 so that

$$x_1 \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix} + x_2 \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} + x_3 \begin{bmatrix} 9 \\ 1 \\ -1 \end{bmatrix} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$$

Make this an augmented matrix and use Gaussian Elimination:

$$\left[\begin{array}{ccc|c} 2 & 1 & 9 & a \\ 1 & 2 & 1 & b \\ 1 & 3 & -1 & c \end{array}\right] \overset{R_1 \leftrightarrow R_2}{\sim} \left[\begin{array}{ccc|c} 1 & 2 & 1 & b \\ 2 & 1 & 9 & a \\ 1 & 3 & -1 & c \end{array}\right]$$

$$\begin{array}{l} -2R_1 + R_2 \to R_2 \\ -1R_1 + R_3 \to R_3 \end{array} \sim \left[\begin{array}{ccc|c} 1 & 2 & 1 & b \\ 0 & -3 & 7 & a-2b \\ 0 & 1 & -2 & c-b \end{array}\right] \overset{R_3 \leftrightarrow R_2}{\sim} \left[\begin{array}{ccc|c} 1 & 2 & 1 & b \\ 0 & 1 & -2 & c-b \\ 0 & -3 & 7 & a-2b \end{array}\right]$$

$$\xrightarrow{3R_2+R_3 \to R_3} \sim \left[\begin{array}{ccc|c} 1 & 2 & 1 & b \\ 0 & 1 & -2 & c-b \\ 0 & 0 & 1 & a-5b+3c \end{array}\right]$$

Then, if we want, we can either convert to a system of equations and back substitute or do Gauss-Jordan Elimination. We'll do the second!

$$\begin{array}{l} 2R_3+R_2 \to R_2 \\ -R_3+R_1 \to R_1 \end{array} \sim \left[\begin{array}{ccc|c} 1 & 2 & 0 & -a+6b-3c \\ 0 & 1 & 0 & 2a-11b+7c \\ 0 & 0 & 1 & a-5b+3c \end{array}\right]$$

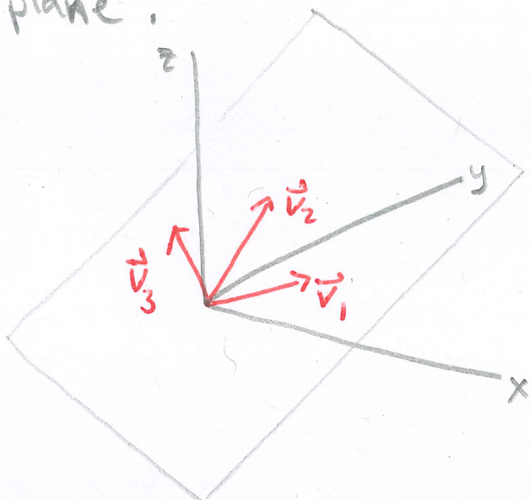
$$-2R_2+R_1 \to R_1 \quad \sim \left[\begin{array}{ccc|c} 1 & 0 & 0 & -5a+28b-17c \\ 0 & 1 & 0 & 2a-11b+7c \\ 0 & 0 & 1 & a-5b+3c \end{array}\right]$$

Now we can see a solution directly. Given any $\vec{v} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$, $\vec{v}$ is in span$\{\vec{u}_1, \vec{u}_2, \vec{u}_3\}$, we need only pick:

$$\begin{cases} x_1 = -5a+28b-17c \\ x_2 = 2a-11b+7c \\ x_3 = a-5b+3c. \end{cases}$$

Question If you take any three vectors in $\mathbb{R}^3$, do you always see that the span is all of $\mathbb{R}^3$?

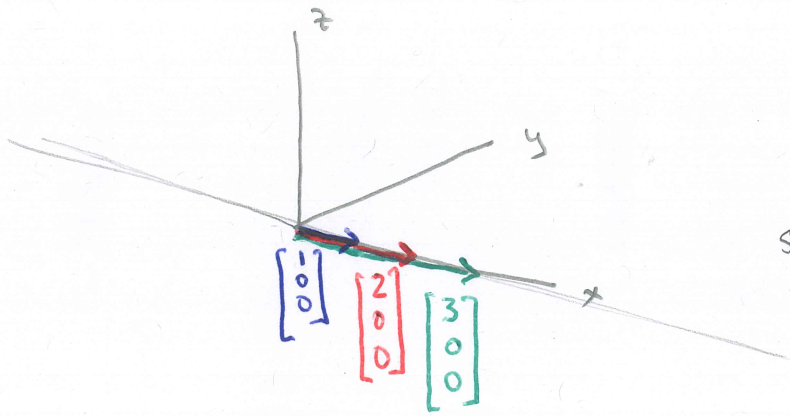
No! You could pick three vectors that lie in the same plane:



This means that there is a linear combination of $\vec{v}_1$ and $\vec{v}_2$ such that $x_1 \vec{v}_1 + x_2 \vec{v}_2 = \vec{v}_3$ (i.e. $\vec{v}_3$ is in the span $\{\vec{v}_1, \vec{v}_2\}$.) You can think of $\vec{v}_3$ as a "redundant" vector, in some sense.

You could also pick three vectors pointing the same direction, with different magnitudes, such as $\begin{bmatrix}1\\0\\0\end{bmatrix}, \begin{bmatrix}2\\0\\0\end{bmatrix}, \begin{bmatrix}3\\0\\0\end{bmatrix}$.
Then span $\left\{\begin{bmatrix}1\\0\\0\end{bmatrix}, \begin{bmatrix}2\\0\\0\end{bmatrix}, \begin{bmatrix}3\\0\\0\end{bmatrix}\right\}$ is a line! Any vector

not pointing along the x-axis is <u>not</u> in the span!



$\text{span}\left\{\begin{bmatrix}1\\0\\0\end{bmatrix}, \begin{bmatrix}2\\0\\0\end{bmatrix}, \begin{bmatrix}3\\0\\0\end{bmatrix}\right\}$ contains any vector along x-axis.

In $\mathbb{R}^n$, these ideas continue to hold. In fact, we can always check if a vector is in the span of another set of vectors in <u>any</u> dimension!

<u>THM</u> Let $\vec{u}_1, \vec{u}_2, \ldots, \vec{u}_m$ and $\vec{v}$ be vectors in $\mathbb{R}^n$. Then $\vec{v}$ is an element of $\text{span}\{\vec{u}_1, \ldots, \vec{u}_m\}$ if and only if the linear system with augmented matrix

$$\left[\begin{array}{cccc|c} \vec{u}_1 & \vec{u}_2 & \cdots & \vec{u}_m & \vec{v} \end{array}\right]$$

has a solution.

<u>REMARK</u> Look back at the previous examples and verify that the theorem above is what we were using. Why? (The why is the proof of this theorem!)

Here's another theorem, motivated by that last question.

THM Let $\vec{u}_1, \vec{u}_2, .., \vec{u}_m$ be vectors in $\mathbb{R}^n$. If $\vec{u}$ is in the span$\{\vec{u}_1, \vec{u}_2, .., \vec{u}_m\}$ (i.e if $\vec{u}$ can be represented as a linear combination of $\vec{u}_1, .., \vec{u}_m$), then

$$\text{span}\{\vec{u}, \vec{u}_1, \vec{u}_2, .., \vec{u}_m\} = \text{span}\{\vec{u}_1, \vec{u}_2, ..., \vec{u}_m\}.$$

Think: "$\vec{u}$ is a 'redundant' vector"

For a rigorous proof, see the textbook (p.70). The proof is just an abstract computation. What is important here is the geometric meaning behind the theorem.

Question When is the span of $\vec{u}_1, \vec{u}_2, .., \vec{u}_m$ all of $\mathbb{R}^n$?

First, we need $m \geq n$, then we need

THM Suppose that $\vec{u}_1, .., \vec{u}_m$ are in $\mathbb{R}^n$ and let

$$A = \begin{bmatrix} \vec{u}_1 & \vec{u}_2 & \cdots & \vec{u}_m \end{bmatrix}$$

be equivalent to a matrix B in echelon form.

Then, span $\{\vec{u}_1, \dots, \vec{u}_m\} = \mathbb{R}^n$ exactly when B has a pivot position in each <u>row</u>.

<u>Question</u>: Why does the theorem imply $m \geq n$? Could you have a pivot position in every row if $m < n$?

<u>EXERCISE</u> Create three matrices, each in echelon form, each with a pivot position in every column. For the first, let $m > n$. For the second, let $m = n$, and for the third, let $m < n$. When do you have one solution? no solutions? Infinitely many? (If you need, make the matrices augmented matrices, where the right column is $\begin{bmatrix} a \\ b \\ c \end{bmatrix}$, for any real numbers a, b, c.)

After doing this exercise, you should notice:

<u>THM</u> Let $\{\vec{u}_1, \vec{u}_2, \dots, \vec{u}_m\}$ be a set of vectors in $\mathbb{R}^n$. If $m < n$, then this set does not span $\mathbb{R}^n$. If $m \geq n$, the set might span $\mathbb{R}^n$ or it might not.

EXERCISE Give your own geometric interpretation of the last two theorems.

Next, we are going to introduce a new notation for a system of equations / linear combination:

Let $A = [\vec{a}_1, \vec{a}_2]$ where $\vec{a}_1 = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}$, $\vec{a}_2 = \begin{bmatrix} 1 \\ 2 \\ 4 \\ 3 \end{bmatrix}$.

Linear combinations of these two vectors look like:

$$x_1 \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix} + x_2 \begin{bmatrix} 1 \\ 2 \\ 4 \\ 3 \end{bmatrix}.$$

Let $\vec{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ and:

$$A\vec{x} = \begin{bmatrix} 1 & 1 \\ 2 & 2 \\ 3 & 4 \\ 4 & 3 \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_1 + x_2 \\ 2x_1 + 2x_2 \\ 3x_1 + 4x_2 \\ 4x_1 + 3x_2 \end{bmatrix}$$

then notice

$$= \begin{bmatrix} x_1 \\ 2x_1 \\ 3x_1 \\ 4x_1 \end{bmatrix} + \begin{bmatrix} x_2 \\ 2x_2 \\ 4x_2 \\ 3x_2 \end{bmatrix} = x_1 \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix} + x_2 \begin{bmatrix} 1 \\ 2 \\ 4 \\ 3 \end{bmatrix},$$

Then,

$A\vec{x}$ is just shorthand for a "linear combination."

In fact, $\boxed{A\vec{x} = \vec{b}}$ is shorthand for a system of equations:

Let $\vec{b} = \begin{bmatrix} 1 \\ 0 \\ 2 \\ 0 \end{bmatrix}$, then $A\vec{x} = \vec{b}$ is the system

$$\begin{bmatrix} 1 & 1 \\ 2 & 2 \\ 3 & 4 \\ 4 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 2 \\ 0 \end{bmatrix}$$

$$x_1 \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix} + x_2 \begin{bmatrix} 1 \\ 2 \\ 4 \\ 3 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 2 \\ 0 \end{bmatrix}$$

which is the same as:

$$\begin{cases} x_1 + x_2 = 1 \\ 2x_1 + 2x_2 = 0 \\ 3x_1 + 4x_2 = 2 \\ 4x_1 + 3x_2 = 0. \end{cases}$$

(this has no solution!)

One of the key ideas is understanding how to multiply a matrix by a vector:

dot the top row with the vector,

$$\begin{bmatrix} \boxed{1 \quad 7} \\ \boxed{3 \quad 6} \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} \boxed{x_1 + 7x_2} \\ \cdots \end{bmatrix}$$

then dot the next row

$$= \begin{bmatrix} x_1 + 7x_2 \\ \boxed{3x_1 + 6x_2} \end{bmatrix}$$

Notice, to "dot" the row with the vector $\vec{x}$, we need the <u>number of colums</u> in the matrix to equal the <u>number of rows</u> in the vector.

With this new notation, we can update our theorems..

<u>THM</u> Let $\vec{a}_1, \vec{a}_2, \ldots, \vec{a}_m$ and $\vec{b}$ be vectors in $\mathbb{R}^n$. Then the following statements are equivalent. (i.e. if one is true, they are <u>all</u> true. If one is false, they are <u>all</u> false).

a) $\vec{b}$ is in span $\{\vec{a}_1, \vec{a}_2, \dots, \vec{a}_m\}$.

b) The vector equation $x_1 \vec{a}_1 + x_2 \vec{a}_2 + \dots + x_m \vec{a}_m = b$ has at least one solution.

c) The linear system corresponding to $[\vec{a}_1 \; \vec{a}_2 \cdots \vec{a}_m \mid \vec{b}]$ has at least one solution.

d) The equation $A\vec{x} = \vec{b}$, where $\vec{A} = [\vec{a}_1 \; \vec{a}_2 \cdots \vec{a}_m]$ and $\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_m \end{bmatrix}$, has at least one solution.

In other words, all of these are different ways to ask the same question!