Lecture #3

2.1 Vectors

DEF A vector is an ordered list of real numbers $u_1, u_2, \ldots, u_n$ expressed as

$$\vec{u} = \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_m \end{bmatrix}$$

or as $\vec{u} = (u_1, u_2, \ldots, u_n)$. The set of all vectors with n entries is denoted by $\mathbb{R}^n$.

The following definitions of vector addition and scalar multiplication should be familiar to you (from M126).

DEF Let $\vec{u}, \vec{v}$ be vectors in $\mathbb{R}^n$ given by

$$\vec{u} = \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{bmatrix} \quad \text{and} \quad \vec{v} = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}.$$

Let c be a real number (a scalar). Then

1. Equality: We say $\vec{u} = \vec{v}$ if and only if $u_1 = v_1, u_2 = v_2, \ldots, u_n = v_n$.

2. Addition: $$\vec{u} + \vec{v} = \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{bmatrix} + \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix} = \begin{bmatrix} u_1 + v_1 \\ u_2 + v_2 \\ \vdots \\ u_n + v_n \end{bmatrix}.$$

③ Scalar multiplication: $c\vec{u} = c\begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{bmatrix} = \begin{bmatrix} cu_1 \\ cu_2 \\ \vdots \\ cu_n \end{bmatrix}$.

The set of all vectors in $\mathbb{R}^n$, taken together with these definitions of addition and scalar multiplication, is called <u>Euclidean Space</u>.

<u>THM</u> (Algebraic Properties of Vectors) Let a, b be scalars, and $\vec{u}, \vec{v}, \vec{w}$ be vectors in $\mathbb{R}^n$. Then

a) $\vec{u} + \vec{v} = \vec{v} + \vec{u}$

b) $a(\vec{u} + \vec{v}) = a\vec{u} + a\vec{v}$

c) $(a+b)\vec{u} = a\vec{u} + b\vec{u}$

d) $(\vec{u} + \vec{v}) + \vec{w} = \vec{u} + (\vec{v} + \vec{w})$

e) $a(b\vec{u}) = (ab)\vec{u}$

f) $\vec{u} + -(\vec{u}) = 0$

g) $\vec{u} + \vec{0} = \vec{0} + \vec{u} = \vec{u}$, $\vec{0} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$ (<u>zero vector</u>).

h) $1\vec{u} = \vec{u}$

<u>EXERCISE</u> Use the definitions of vector addition and scalar multiplication to show that the statements above are true.

Now, let's ask a simple question. Given the vectors $\begin{bmatrix}1\\0\end{bmatrix}$ and $\begin{bmatrix}0\\1\end{bmatrix}$, can I <u>combine</u> these vectors in some way to get the vector $\begin{bmatrix}5\\6\end{bmatrix}$? As in, can I multiply each vector by a (possibly) different number and add them up to get $\begin{bmatrix}5\\6\end{bmatrix}$?

<u>Yes!</u> Multiply the first vector by 5 and the second by 6, add them up, and:

$$5\begin{bmatrix}1\\0\end{bmatrix} + 6\begin{bmatrix}0\\1\end{bmatrix} = \begin{bmatrix}5\\6\end{bmatrix}.$$

We are going to be interested in equations of this form, so we'll give a definition.

<u>DEF</u> If $\vec{u}_1, \vec{u}_2, \ldots, \vec{u}_m$ are vectors and $c_1, \ldots, c_m$ are scalars, then

$$c_1\vec{u}_1 + c_2\vec{u}_2 + \cdots + c_m\vec{u}_m$$

is a <u>linear combination</u> of $\vec{u}_1, \vec{u}_2, \ldots, \vec{u}_m$.

Note that we allow scalars here to be negative or equal to zero, not just positive!

Okay, so we have a new thing, a vector, and we can consider liner combinations of vectors, but what does that have to do with systems of equations studied in the first chapter?

Notice the following: let x_1, x_2, and x_3 be real numbers and consider

$$x_1 \begin{bmatrix} 1 \\ 2 \\ -4 \end{bmatrix} + x_2 \begin{bmatrix} -3 \\ -5 \\ 13 \end{bmatrix} + x_3 \begin{bmatrix} 2 \\ -1 \\ -12 \end{bmatrix} = \begin{bmatrix} -1 \\ 2 \\ 11 \end{bmatrix}.$$

The is the same thing as the following system of equations!

$$\begin{cases} x_1 - 3x_2 + 2x_3 = -1 \\ 2x_1 - 5x_2 - x_3 = 2 \\ -4x_1 + 13x_2 - 12x_3 = 11 \end{cases}.$$

Recall, this was <u>EXAMPLE 1</u> in <u>Lecture #2</u>, and we saw that the solution was $x_1 = 50$, $x_2 = 19$, and $x_1 = 3$.

In other words, solving that system of equations is also answering the following question: Is there a liner combination of the vectors $\begin{bmatrix} 1 \\ 2 \\ -4 \end{bmatrix}$, $\begin{bmatrix} -3 \\ -5 \\ 13 \end{bmatrix}$, and $\begin{bmatrix} 2 \\ -1 \\ -12 \end{bmatrix}$ that sums to the vector $\begin{bmatrix} -1 \\ 2 \\ 11 \end{bmatrix}$.

Let's try an example.

<u>EXAMPLE 1</u>

A bag of Vigoro fertilizer contains 29 pounds of nitrogen, 3 pounds of phosphoric acid, and 4 pounds of potash. A bag of Parker's fertilizer contains 18 pounds of nitrogen, 25 pounds of phosphoric acid, and 6 pounds of potash.

How many bags of Vigoro and Parker's fertilizer should you combine so that you have a mixture containing 148 pounds of nitrogen, 131 pounds of phosphoric acid, and 38 pounds of potash?

Let's make vectors representing bags of Vigoro and Parker's fertilizer.

$$\text{Vigoro:} \quad \vec{v} = \begin{bmatrix} 29 \\ 3 \\ 4 \end{bmatrix} \begin{array}{l} \text{pounds of nitrogen} \\ \text{pounds of phosphoric acid} \\ \text{pounds of potash.} \end{array}$$

$$\text{Parker's:} \quad \vec{p} = \begin{bmatrix} 18 \\ 25 \\ 6 \end{bmatrix} \begin{array}{l} \text{pounds of nitrogen} \\ \text{pounds of phosphoric acid} \\ \text{pounds of potash} \end{array}$$

Now, we can rephrase the question. We would like to find x_1, the number of bags of Vigoro, and x_2, the number of bags of Parker's, such that

$$x_1 \overset{\vec{V}}{\begin{bmatrix} 29 \\ 3 \\ 4 \end{bmatrix}} + x_2 \overset{\vec{P}}{\begin{bmatrix} 18 \\ 25 \\ 6 \end{bmatrix}} = \begin{bmatrix} 148 \\ 131 \\ 38 \end{bmatrix}.$$

This is the same as solving the system of equations:

$$\begin{cases} 29x_1 + 18x_2 = 148 \\ 3x_1 + 25x_2 = 131 \\ 4x_1 + 6x_2 = 38 \end{cases}.$$

We use Gaussian Elimination (notice how the vectors $\vec{V}$ and $\vec{P}$ are in the augmented matrix!).

$$\begin{array}{c} \begin{array}{ccc} \vec{V} & \vec{P} & \end{array} \\ \left[\begin{array}{cc|c} 29 & 18 & 148 \\ 3 & 25 & 131 \\ 4 & 6 & 38 \end{array}\right] \end{array}$$

$$\sim \left[\begin{array}{cc|c} 4 & 6 & 38 \\ 0 & 1 & 5 \\ 0 & 0 & 0 \end{array}\right]$$

<u>EXERCISE</u>
Perform the following row operations!

$R_1 \longleftrightarrow R_2$

$-\frac{3}{4}R_1 + R_2 \longrightarrow R_2$

$-\frac{29}{4}R_1 + R_3 \longrightarrow R_3$

$\frac{51}{41}R_2 + R_3 \longrightarrow R_3$

$\frac{2}{41}R_2 \longrightarrow R_2$

Turn it back into a system of equations and use back substitution.

$$4x_1 + 6x_2 = 38$$
$$x_2 = 5$$

$$\Rightarrow 4x_1 + 6(5) = 38$$
$$4x_1 = 8$$
$$x_1 = 2.$$

So, we need two bags of Vigoro and five bags of Parker's.

$$\boxed{\text{Note, we can express our solution as a vector!}}$$

$$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 2 \\ 5 \end{bmatrix}.$$

EXERCISE Following this example, can you make a mixture containing 147 pounds of nitrogen, 84 pounds of phosphoric acid, and 30 pounds of potash? (Hint: No...)

Q: What does that mean about the vectors $\vec{v}$ and $\vec{p}$?

EXAMPLE 2

Recall EXAMPLE 3 from Lecture #2. Express all solutions the system of linear equations

$$\begin{cases} 6x_3 + 19x_5 + 11x_6 = -27 \\ 3x_1 + 12x_2 + 9x_3 - 6x_4 + 26x_5 + 31x_6 = -63 \\ x_1 + 4x_2 + 3x_3 - 2x_4 + 10x_5 + 9x_6 = -17 \\ -x_1 - 4x_2 - 4x_3 + 2x_4 - 13x_5 - 11x_6 = 22. \end{cases}$$

Recall, the solution was

$$\begin{aligned} x_1 &= -5 - 4s_3 + 2s_2 - 4s_1 \\ x_2 &= s_3 \\ x_3 &= -14 - 5s_1 \\ x_4 &= s_2 \\ x_5 &= 3 + s_1 \\ x_6 &= s_1. \end{aligned}$$

In vector form:

$$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{bmatrix} = \begin{bmatrix} -5 - 4s_3 + 2s_2 - 4s_1 \\ s_3 \\ -14 - 5s_1 \\ s_2 \\ 3 + s_1 \\ s_1 \end{bmatrix} =$$

(pull off the constant terms...)

$$= \underset{\text{constants}}{\begin{bmatrix}-5\\0\\-14\\0\\3\\0\end{bmatrix}} + \begin{bmatrix}-4s_3+2s_2-4s_1\\ s_3\\ -5s_1\\ s_2\\ s_1\\ s_1\end{bmatrix}$$

$$= \underset{\text{constants}}{\begin{bmatrix}-5\\0\\-14\\0\\3\\0\end{bmatrix}} + \underset{s_1\text{ terms}}{\begin{bmatrix}-4s_1\\0\\-5s_1\\0\\s_1\\s_1\end{bmatrix}} + \underset{s_2\text{ terms}}{\begin{bmatrix}2s_2\\0\\0\\s_2\\0\\0\end{bmatrix}} + \underset{s_3\text{ terms}}{\begin{bmatrix}-4s_3\\s_3\\0\\0\\0\\0\end{bmatrix}}$$

pull out the scalar parameters

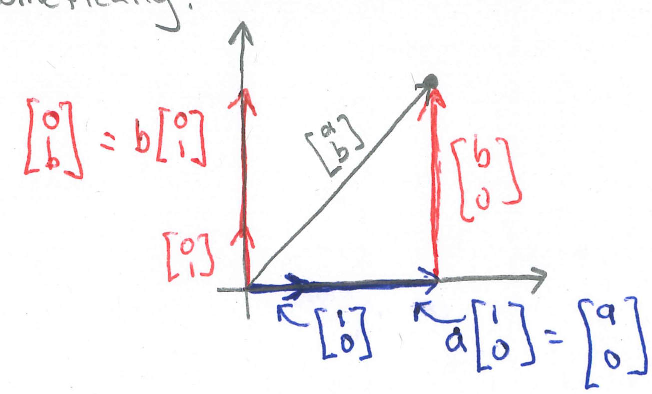
$$= \underbrace{\begin{bmatrix}-5\\0\\-14\\0\\3\\0\end{bmatrix} + s_1\begin{bmatrix}-4\\0\\-5\\0\\1\\1\end{bmatrix} + s_2\begin{bmatrix}2\\0\\0\\1\\0\\0\end{bmatrix} + s_3\begin{bmatrix}-4\\1\\0\\0\\0\\0\end{bmatrix}}_{\text{Vector form of solution.}}.$$

Next, let's answer a series of simpler questions and try to develop some geometric intuition for when solutions exist, when they don't exist, and when there are infinitely many. (This will help you answer the additional question following the exercise before EXAMPLE 2.)

<u>Question 1</u>: Given the vectors $\begin{bmatrix}1\\0\end{bmatrix}$ and $\begin{bmatrix}0\\1\end{bmatrix}$, is there a linear combination equal to $\begin{bmatrix}a\\b\end{bmatrix}$, where a and b are <u>any</u> real numbers?

Yes!
$$a\begin{bmatrix}1\\0\end{bmatrix} + b\begin{bmatrix}0\\1\end{bmatrix} = \begin{bmatrix}a\\b\end{bmatrix}$$

Geometrically:



i.e., we can get to any place in $\mathbb{R}^2$ using these two vectors.

Question 2: Given the vectors $\begin{bmatrix}1\\0\end{bmatrix}$ and $\begin{bmatrix}2\\0\end{bmatrix}$, is there a linear combination equal to $\begin{bmatrix}a\\b\end{bmatrix}$, where a and b are any real numbers?

No! Try to find x_1 and x_2 such that

$$x_1\begin{bmatrix}1\\0\end{bmatrix} + x_2\begin{bmatrix}2\\0\end{bmatrix} = \begin{bmatrix}a\\b\end{bmatrix}$$

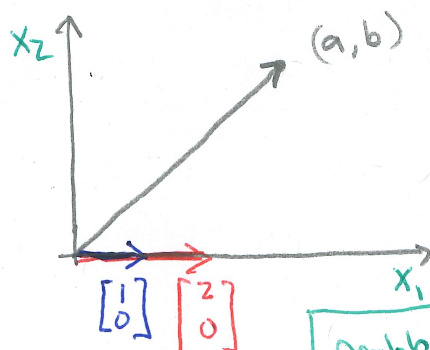
$$\begin{bmatrix}x_1\\0\end{bmatrix} + \begin{bmatrix}2x_2\\0\end{bmatrix} = \begin{bmatrix}a\\b\end{bmatrix}$$

$$\begin{bmatrix}x_1+2x_2\\0\end{bmatrix} = \begin{bmatrix}a\\b\end{bmatrix}$$

and we see there is no solution. In terms of a system of equations, we have

$$\begin{cases} x_1 + 2x_2 = a \\ 0 = b. \end{cases}$$

Geometrically,



Linear combinations of $\begin{bmatrix}1\\0\end{bmatrix}$ and $\begin{bmatrix}2\\0\end{bmatrix}$ can only give you vectors along x_1-axis!

$\boxed{\text{problem: they are parallel!}}$ (*)

EXERCISE Without actually solving, which of the sets of vectors below permit a linear combination equal to $\begin{bmatrix} a \\ b \end{bmatrix}$ for *any* a, b real numbers? Is it unique?

1. $\left\{ \begin{bmatrix} 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 3 \end{bmatrix} \right\}$

2. $\left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 5 \\ 5 \end{bmatrix} \right\}$

3. $\left\{ \begin{bmatrix} 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 10 \\ 0 \end{bmatrix} \right\}$

4. $\left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ -1 \end{bmatrix} \right\}$

5. $\left\{ \begin{bmatrix} 3 \\ 15 \end{bmatrix}, \begin{bmatrix} -6 \\ -10 \end{bmatrix} \right\}$.

6. $\left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right\}$.