Lecture #2

1.2 Linear Systems and Matrices

Our goal in this lecture is to learn how to take a general system of linear equations and turn it into (ideally) a triangular system. This won't always work, but we will see that we can always turn it into an echelon system.

We will then try to simplify this procedure, distilling only the important elements, which will give us a reason to introduce matrices.

We begin by claiming the following. The two systems of linear equations below are equivalent, meaning they have the same solution set.

$$\begin{cases} x_1 - 3x_2 + 2x_2 = -1 \\ 2x_1 - 5x_2 - x_3 = 2 \\ -4x_1 + 13x_2 - 12x_3 = 11 \end{cases} \overset{\text{"equivalent"}}{\sim} \begin{cases} x_1 - 3x_2 + 2x_3 = -1 \\ x_2 - 5x_3 = 4 \\ x_3 = 3 \end{cases}$$

To show these two systems are "the same," we need to come up with a set of modifications we can make to a system of equations that doesn't change the solution.

We call these $\boxed{\text{elementary operations.}}$

Elementary Operations

1) We can $\boxed{\text{interchange the position of two equations.}}$

E.g.

$$\begin{cases} 3x_1 - 5x_2 - 8x_3 = -4 \\ x_1 + 2x_2 - 4x_3 = 5 \\ -2x_1 + 6x_2 + x_3 = 3 \end{cases} \sim \begin{cases} x_1 + 2x_2 - 4x_3 = 5 \\ 3x_1 - 5x_2 - 8x_3 = -4 \\ -2x_1 + 6x_2 + x_3 = 3 \end{cases}$$

① ⟷ ② ⟵ shorthand notation for the operation.

(interchanged the first equation with the second)

This doesn't change the solution set!

2) We can $\boxed{\text{multiply an equation by a nonzero constant.}}$

E.g.

$$\begin{cases} x_1 + 2x_2 - 4x_3 = 5 \\ 3x_1 - 5x_2 - 8x_3 = -4 \\ -2x_1 + 6x_2 + x_3 = 3 \end{cases} \sim \begin{cases} x_1 + 2x_2 - 4x_3 = 5 \\ 3x_1 - 5x_2 - 8x_3 = -4 \\ 4x_1 - 12x_2 - 2x_3 = 3 \end{cases}$$

$-2 \cdot ③ \longrightarrow ③$

(multiply equation 3 by -2, and use that in place of equation 3)

3) We can $\boxed{\text{add a multiple of one equation to another.}}$

$$\begin{cases} x_1 + 2x_2 - 4x_3 = 5 \\ 3x_1 - 5x_2 - 8x_3 = -4 \\ 4x_1 - 12x_2 - 2x_3 = -6 \end{cases} \sim \begin{cases} x_1 + 2x_2 - 4x_3 = 5 \\ 3x_1 - 5x_2 - 8x_3 = -4 \\ -20x_2 + 14x_3 = -26. \end{cases}$$

$$-4 \cdot (1) + (3) \longrightarrow (3)$$

(multiply equation 1 by -4, then add that to equation 3. Then replace equation 3 with the result.)

These are the only three operations you need!

<u>EXERCISE</u> Explain why none of the elementary operations change the solution to the system. Do any of the operations introduce <u>new</u> constraints on the system? Do any remove existing constraints?

<u>EXAMPLE 1</u>

Find the set of solutions to the system of linear equations

$$\begin{cases} x_1 - 3x_2 + 2x_3 = -1 \\ 2x_1 - 5x_2 - x_3 = 2 \\ -4x_1 + 13x_2 - 12x_3 = 11 \end{cases}$$

We will do this by using the elementary operations.

Our first step will be to eliminate x_1 from the second and third equation.

First, use the third elementary operation: multiply equation ① by -2, then add it to equation ②. Replace equation ② with the result.

$$\begin{cases} x_1 - 3x_2 + 2x_3 = -1 \\ 2x_1 - 5x_2 - x_3 = 2 \\ -4x_1 + 13x_2 - 12x_3 = 11 \end{cases} \sim \begin{cases} x_1 - 3x_2 + 2x_3 = -1 \\ \quad\;\; x_2 - 5x_3 = 4 \\ -4x_1 + 13x_2 - 12x_3 = 11 \end{cases}$$

$$\underbrace{-2\,① + ②}_{} \longrightarrow ②$$

$$\begin{array}{r} -2x_1 + 6x_2 - 4x_3 = 2 \\ +\ (2x_1 - 5x_2 - x_3 = 2) \\ \hline 0 + x_2 - 5x_3 = 4 \end{array}$$

Next, use the third elementary operation again. Multiply equation ① by 4 and add it to equation ③. Replace equation ③ with the result.

$\downarrow$

$$\begin{cases} x_1 - 3x_2 + 2x_3 = -1 \\ x_2 - 5x_3 = 4 \\ -4x_1 + 13x_2 - 12x_3 = 11 \end{cases} \sim \begin{cases} x_1 - 3x_2 + 2x_3 = -1 \\ x_2 - 5x_3 = 4 \\ x_2 - 4x_3 = 7 \end{cases}$$

$$\underbrace{4 \cdot 1 + 3} \longrightarrow 3$$

$$\begin{array}{r} 4x_1 - 12x_2 + 8x_3 = -4 \\ + \ (-4x_1 + 13x_2 - 12x_3 = 11) \\ \hline x_2 - 4x_3 = 7 \end{array}$$

Finally, we'll use the third operation again to turn the system into a triangular system. Yes, the third elementary operation is important! Also, notice that equations 2 and 3 are now a system of equations with two variables.

$$\begin{cases} x_1 - 3x_2 + 2x_3 = -1 \\ x_2 - 5x_3 = 4 \\ x_2 - 4x_3 = 7 \end{cases} \sim \begin{cases} x_1 - 3x_2 + 2x_3 = -1 \\ x_2 - 5x_3 = 4 \\ x_3 = 3 \end{cases}$$

$$\underbrace{-1 \cdot 2 + 3} \longrightarrow 3$$

$$\begin{array}{r} -x_2 + 5x_3 = -4 \\ + \ (x_2 - 4x_3 = 7) \\ \hline x_3 = 3 \end{array}$$

(Notice that this proves the claim on the first page of the notes!)

Now, we can use back substitution to solve.

$x_3 = 3$, plug into ②: $\quad x_2 - 5(3) = 4$

$$x_2 = 19$$

$x_3 = 3$, $x_2 =$, plug into ①: $\quad x_1 - 3(19) + 2(3) = -1$

$$x_1 = 50$$

So, we get
$$\begin{cases} x_1 = 50 \\ x_2 = 19 \\ x_3 = 3 \end{cases}.$$

<u>REMARK</u> You can always check your answer by plugging into each equation!

Next, notice that we are doing a lot of writing... and some of it is unnecessary. Notice that if in each equation, we keep our variables ordered from left to right, we are really only modifying the coefficents. We can actually create a system that keeps track of these coefficients, making our computations a little "lighter."

To do this, we will define a special kind of <u>matrix</u>, just a rectangular table of numbers. We will call it an <u>augmented matrix</u>.

Take a linear system, and distill only the important elements: the coefficients. For example:

<u>Linear System</u>   <u>Augmented Matrix</u>

coeff. for 1st eq.

$$\begin{cases} x_1 - 3x_2 + 2x_3 = 1 \\ 2x_1 - 5x_2 - x_3 = 2 \\ -4x_1 + 13x_2 - 12x_3 = 11 \end{cases} \rightsquigarrow \left[\begin{array}{ccc|c} 1 & -3 & 2 & 1 \\ 2 & -5 & -1 & 2 \\ -4 & 13 & -12 & 11 \end{array}\right]$$

coeff. for 3rd equation

coefficients in front of x_1

line denotes "equal" sign

In general, this looks like:

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = b_2 \\ \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n = b_m \end{cases} \rightsquigarrow \left[\begin{array}{cccc|c} a_{11} & a_{12} & \cdots & a_{1n} & b_1 \\ a_{12} & a_{22} & \cdots & a_{2n} & b_2 \\ & \vdots & & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} & b_m \end{array}\right]$$

In this new context (augmented matrix), we replace the elementary operations (for linear systems) with <u>elementary row operations</u>.

<u>Elementary Row Operations</u>

1) We can interchange two rows.

2) We can multiply a row by a nonzero constant.

3) We can replace a row with the sum of that row and a scalar multiple of another row.

<u>EXERCISE</u> Compare the Elementary Operations with the Elementary Row Operations. Is it clear these operations are the same?

<u>EXAMPLE 2</u>

Find all solutions to the system of linear equations

$$\begin{cases} 2x_1 - 3x_2 + 10x_3 = -2 \\ x_1 - 2x_2 + 3x_3 = -2 \\ -x_1 + 3x_2 + x_3 = 4 \end{cases}.$$

Turn the system into an augmented matrix and work with that:

$$\left[\begin{array}{ccc|c} 2 & -3 & 10 & -2 \\ 1 & -2 & 3 & -2 \\ -1 & 3 & 1 & 4 \end{array}\right]$$

One is a very nice coefficient ... so let's make the second equation the first equation. Then, at the end, when we solve for x_1, we won't have to divide by a coefficient. Additionally, this will make it easier to "eliminate" the x_1 variable in the other equations. (Here, that means make the coefficient 0.).

$$R_1 \leftrightarrow R_2 \quad \sim \left[\begin{array}{ccc|c} 1 & -2 & 3 & -2 \\ 2 & -3 & 10 & -2 \\ -1 & 3 & 1 & 4 \end{array}\right]$$

new shorthand! (R_1 denotes row 1, etc.)

Now, multiply the top row by -2 and add it to the second:

$$-2 \cdot R_1 + R_2 \rightarrow R_2 \quad \sim \left[\begin{array}{ccc|c} 1 & -2 & 3 & -2 \\ 0 & 1 & 4 & 2 \\ -1 & 3 & 1 & 4 \end{array}\right]$$

Next, add row 1 to row 3:

$R_1 + R_3 \to R_3$

$$\sim \left[\begin{array}{ccc|c} 1 & -2 & 3 & -2 \\ 0 & 1 & 4 & 2 \\ 0 & 1 & 4 & 2 \end{array}\right].$$

Now, multiply row 2 by negative 1 and add it to row 3.

$-1 \cdot R_2 + R_3 \to R_3$

$$\sim \left[\begin{array}{ccc|c} 1 & -2 & 3 & -2 \\ 0 & 1 & 4 & 2 \\ 0 & 0 & 0 & 0 \end{array}\right].$$

Notice, the last row represents the equation $\underline{0=0}$, i.e., we had a redundant equation somewhere!

This means that our system of equations has become

$$\begin{cases} x_1 - 2x_2 + 3x_3 = -2 \\ \quad x_2 + 4x_3 = 2 \\ \quad 0 = 0. \end{cases}$$

This is now in echelon form, and we see that x_3 is a free variable. So, let $x_3 = s_1$, a parameter, and solve for x_2 (back substitute):

$$x_2 + 4s_1 = 2$$
$$x_2 = 2 - 4s_1.$$

Now, solve for x_1.

$$x_1 - 2(2-4s_1) + 3s_1 = -2$$
$$x_1 - 4 + 8s_1 + 3s_1 = -2$$
$$x_1 = 2 - 11s_1$$

So our solution is:

$$\begin{cases} x_1 = 2 - 11s_1 \\ x_2 = 2 - 4s_1 \\ x_3 = s_1 \end{cases}$$

The procedure that we used in this example is called $\boxed{\text{\underline{Gaussian} \underline{Elimination}.}}$

Also, notice that since each augmented matrix corresponds to a system of linear equations, we can say what we mean by a matrix being in echelon form:

DEF We say that an augmented matrix is in $\boxed{\text{echelon form}}$ if the corresponding system of equations is in echelon form. Another way to say this is that an augmented matrix is in echelon form if

1) every $\boxed{\text{leading term}}$ (the first non-zero term in a row) is in a column to the left of the leading term of the row below it, and

2) any zero rows are at the bottom of the matrix.

EXERCISE Compare the two versions of this definition. Can you prove that they are equivalent?

We will do another example using Gaussian Elimination, but first, we are going to refine our technique. Notice how the first step in the last example was to pick a "nice" equation to be in the top row. Two reasons it was "nice": ① it had a nonzero coefficient for x_1 and ② the coefficient was 1. This choice amounts to picking a $\boxed{\text{pivot position}}$ for the first row.

DEF If you have a matrix in echelon form, the $\boxed{\text{pivot positions}}$ are the positions in the matrix containing a leading term.

A $\boxed{\text{pivot}}$ is a non-zero number in a pivot position. A $\boxed{\text{pivot column}}$ is a column in a matrix that contains a pivot position.

Take the matrix in echelon form from the last example.

these are pivot positions (both pivots are 1.)

$$\left[\begin{array}{ccc|c} \boxed{1} & -2 & 3 & -2 \\ 0 & \boxed{1} & 4 & 2 \\ 0 & 0 & 0 & 0 \end{array}\right].$$

these are pivot columns

Now, let's do another example using Gaussian Elimination, but this time, we'll use our new terminology.

EXAMPLE 3

Find all solutions to the system of linear equations

$$\begin{cases} 6x_3 + 19x_5 + 11x_6 = -27 \\ 3x_1 + 12x_2 + 9x_3 - 6x_4 + 26x_5 + 31x_6 = -63 \\ x_1 + 4x_2 + 3x_3 - 2x_4 + 10x_5 + 9x_6 = -17 \\ -x_1 - 4x_2 - 4x_3 + 2x_4 - 13x_5 - 11x_6 = 22 \end{cases}$$

Start by transforming this system into a matrix.

$$\left[\begin{array}{cccccc|c} 0 & 0 & 6 & 0 & 19 & 11 & -27 \\ 3 & 12 & 9 & -6 & 26 & 31 & -63 \\ 1 & 4 & 3 & -2 & 10 & 9 & -17 \\ -1 & -4 & -4 & 2 & -13 & -11 & 22 \end{array}\right]$$

Now, $\boxed{\text{pick a pivot position for the first row}}$ (and the pivot, i.e. the coefficient we want). Notice, "a_{11}", the top left corner, should be our pivot position since there are non-zero entries in the first column. The third row looks like a great option (notice the 1).

Perform a row operation:

$$R_1 \leftrightarrow R_3 \quad \sim \left[\begin{array}{cccccc|c} 1 & 4 & 3 & -2 & 10 & 9 & -17 \\ 3 & 12 & 9 & -6 & 26 & 31 & -63 \\ 0 & 0 & 6 & 0 & 19 & 11 & -27 \\ -1 & -4 & -4 & 2 & -13 & -11 & 22 \end{array}\right]$$

Now, eliminate the coefficients in the first column beneath row one. We'll perform two row operations to accomplish this:

$$\begin{array}{l} -3 \cdot R_1 + R_2 \rightarrow R_2 \\ R_1 + R_4 \rightarrow R_4 \\ \sim \end{array} \left[\begin{array}{cccccc|c} 1 & 4 & 3 & -2 & 10 & 9 & -17 \\ 0 & 0 & 0 & 0 & -4 & 4 & -12 \\ 0 & 0 & 6 & 0 & 19 & 11 & -27 \\ 0 & 0 & -1 & 0 & -3 & -2 & 5 \end{array}\right]$$

And that is basically it for Gaussian Elimination, two steps. Now, we just repeat the process, looking at the second row instead of the first.

Pick a pivot position for the second row. Notice that none of the remaining rows have a non-zero number in the second column, so our pivot position cannot be the second spot. But it can be the third spot.

Let's go ahead and move the fourth row up to the second:

$$R_2 \leftrightarrow R_4 \quad \sim \left[\begin{array}{cccccc|c} 1 & 4 & 3 & -2 & 10 & 9 & -17 \\ 0 & 0 & -1 & 0 & -3 & -2 & 5 \\ 0 & 0 & 6 & 0 & 19 & 11 & -27 \\ 0 & 0 & 0 & 0 & -4 & 4 & -12 \end{array}\right]$$

And now, we $\boxed{\text{eliminate}}$ any non-zero coefficient below the pivot position of the second row. We'll only need to do one row operation this time.

$$6 \cdot R_2 + R_3 \rightarrow R_3 \quad \sim \left[\begin{array}{cccccc|c} 1 & 4 & 3 & -2 & 10 & 9 & -17 \\ 0 & 0 & -1 & 0 & -3 & -2 & 5 \\ 0 & 0 & 0 & 0 & 1 & -1 & 3 \\ 0 & 0 & 0 & 0 & -4 & 4 & -12 \end{array}\right]$$

Now, repeat again. $\boxed{\text{Pick a pivot position for the third row.}}$ Notice, the fourth column cannot be a pivot position, but the fifth can. In fact, we don't even need to use a row operation. The matrix looks good as is.

Next, $\boxed{\text{eliminate}}$ any non-zero coefficient below the pivot position of the third row.

We'll need to do one more row operation to do this.

$$4 \cdot R_3 + R_4 \to R_4 \quad \sim \left[\begin{array}{cccccc|c} 1 & 4 & 3 & -2 & 10 & 9 & -17 \\ 0 & 0 & -1 & 0 & -3 & -2 & 5 \\ 0 & 0 & 0 & 0 & 1 & -1 & 3 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right].$$

(EXERCISE Identify the pivot positions, pivots, and pivot columns in the augmented matrix above.)

Now, the matrix is in echelon form, so we can solve. Turn it back into a system of equations.

$$\begin{cases} x_1 + 4x_2 + 3x_3 - 2x_4 + 10x_5 + 9x_6 = -17 \\ \quad -x_3 \quad - 3x_5 - 2x_6 = 5 \\ \quad x_5 - x_6 = 3 \\ \quad 0 = 0 \end{cases}$$

Notice, x_6 is a free variable, so let $x_6 = s_1$, a parameter. Then back substitute: $x_5 - s_1 = 3$, so $x_5 = s_1 + 3$. Then plug into equation two,

$$-x_3 - 3(s_1+3) - 2s_1 = 5$$

$$x_3 = -14 - 5s_1.$$

Then, notice that both x_4 and x_2 are also free variables, so let $x_4 = s_2$ and $x_2 = s_3$, where s_1 and s_2 are parameters (any real number). Then plug into equation one and solve for x_1.

$$x_1 + 4s_3 + 3(-14 - 5s_1) - 2s_2 + 10(s_1 + 3) + 9s_1 = -17$$

$$x_1 = -5 - 4s_3 + 2s_2 - 4s_1 .$$

Then, the solution is

$$\begin{cases} x_1 = -5 - 4s_3 + 2s_2 - 4s_1 \\ x_2 = s_3 \\ x_3 = -14 - 5s_1 \\ x_4 = s_2 \\ x_5 = 3 + s_1 \\ x_6 = s_1 . \end{cases}$$

EXERCISE

Find all solutions to $\begin{cases} x_1 + 4x_2 - 3x_3 = 2 \\ 3x_1 - 2x_2 - x_3 = -1 \\ -x_1 + 10x_2 - 5x_3 = 3 . \end{cases}$

[Hint: there aren't any...]

Next, we are going to make one more improvement to our procedure. Notice how at the end of every example, we turn the matrix back into a system of equations and do more arithmetic. We are going to add one more step that will help minimize the amount of arithmetic.

Let's go back to the matrix in echelon form from the previous example and make the following changes:

1) Make 1 the leading term in each row. (Multiply the row by the reciprocal of the pivot.)

2) Use row operations to make the entries *above* each pivot position 0.

Doing Gaussian Elimination followed by these two steps is called **Gauss - Jordan Elimination.**

The augmented matrix we get at the end is a matrix in **reduced echelon form.**

Here's that matrix in echelon form:

$$\left[\begin{array}{cccccc|c} 1 & 4 & 3 & -2 & 10 & 9 & -17 \\ 0 & 0 & -1 & 0 & -3 & -2 & 5 \\ 0 & 0 & 0 & 0 & 1 & -1 & 3 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

Now, let's do step 1. The only leading term that's not 1 is the -1 in the second row. Perform a row operation to fix this.

$$-1 \cdot R_2 \to R_2 \quad \sim \left[\begin{array}{cccccc|c} 1 & 4 & 3 & -2 & 10 & 9 & -17 \\ 0 & 0 & 1 & 0 & 3 & 2 & -5 \\ 0 & 0 & 0 & 0 & 1 & -1 & 3 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

Now, we look at the leading term in the third row, and perform row operations so that each entry in the same column above the leading term becomes 0!

$$\begin{array}{c} -10R_3 + R_1 \to R_1 \\ -3R_3 + R_2 \to R_2 \end{array} \quad \sim \left[\begin{array}{cccccc|c} 1 & 4 & 3 & -2 & 0 & 19 & -47 \\ 0 & 0 & 1 & 0 & 0 & 5 & -14 \\ 0 & 0 & 0 & 0 & 1 & -1 & 3 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

Next, we need the entry above the leading term in the second row to be zero. With one more operation:

$$-3R_2+R_1 \to R_1 \quad \sim \left[\begin{array}{cccccc|c} 1 & 4 & 0 & -2 & 0 & 4 & -5 \\ 0 & 0 & 1 & 0 & 0 & 5 & -14 \\ 0 & 0 & 0 & 0 & 1 & -1 & 3 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right],$$

and we are done! The matrix is in <u>reduced echelon form</u>. And, it is not too hard to see which variables are free, and what the solution looks like. (The leading variables are not free, but the others are.) Let $x_6 = s_1$, $x_4 = s_2$, and $x_2 = s_3$. Then the matrix says:

$$x_1 + 4s_3 - 2s_2 + 4s_6 = -5$$

$$x_3 + 5s_1 = -14$$

$$x_5 - s_1 = 3,$$

which quickly becomes the solution that we saw in the last example.

In fact, if the echelon form of the system is actually a triangular form, then the solution is even easier to see in matrix form. Try the following exercise.

EXERCISE

Use Gauss-Jordan elimination to find all solutions to the system of linear equations.

$$\begin{cases} x_1 - 2x_2 - 3x_3 = -1 \\ x_1 - x_2 - 2x_3 = 1 \\ -x_1 + 3x_2 + 5x_3 = 2. \end{cases}$$

[Hint: You should get $x_1 = 2$, $x_2 = 3$, $x_3 = -1$.]

There is a really cool theorem about reduced echelon form, which explains its close relationship to the solution (and, hence, why we are so interested in it).

THM

A given matrix is equivalent to a $\boxed{\text{unique}}$ matrix that is in reduced echelon form.

We won't give a proof of this, but if you are interested, talk to me!

There is one more theorem that is incredibly important, which validates some of our intuition about linear systems.

THM

A system of linear equations has no solutions, exactly one solution, or infinitely many solutions.

pf: Take any system, turn it into an augmented matrix, use Gaussian Elimination, turn it back into an (echelon) system of equations.

There are only three things that can happen. Either you have an equation of the form $0 = c$, $(c \neq 0)$, so there are no solutions, or one of the following happens:

1) Every variable is a leading variable, so the echelon system is a triangular system, and there is exactly one solution, or

2) You have at least one free variable, in which case there are infinitely many solutions.

DEF

We need to define one last thing, a homogeneous linear system. This is a system where $b_1 = b_2 = \cdots = b_m = 0$, i.e. it is of the form:

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = 0 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = 0 \\ \quad \vdots \qquad\qquad\qquad\qquad \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n = 0 \end{cases}$$

<u>Notice</u> There is always at least one solution! $x_1 = x_2 = x_3 = \cdots = x_n = 0$.

If there are other solutions, we call them <u>non-trivial solutions</u>.

The solution $x_1 = x_2 = x_3 = \cdots = x_n = 0$ is the <u>trivial solution</u>.

<u>EXERCISE</u>

Review and complete the conceptual problems for chapter 1 found on the course website under "Other Materials."