Lecture #1

1.1 Lines and Linear Equations

Recall how to solve a system of equations.

EXAMPLE 1

$$\begin{cases} ① & x_1 + x_2 = 100 \\ ② & 3x_1 + 5x_2 = 422 \end{cases}$$

To solve, take "3 times equation ①" and subtract equation ②. This eliminates the x_1 variable, and enables us to solve for x_2:

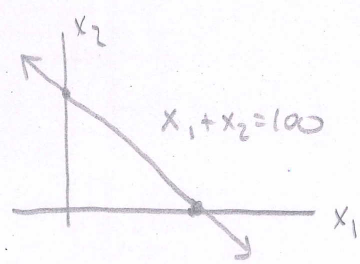
$$3\cdot① - ②: \quad \begin{array}{r} 3x_1 + 3x_2 = 300 \\ -(3x_1 + 5x_2 = 422) \\ \hline 0 + {}^{-}2x_2 = -122 \end{array} \rightsquigarrow x_2 = 61.$$

Then, plug $x_2 = 61$ into either ① or ② to get x_1.

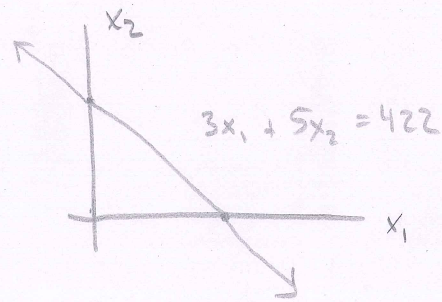
$$x_1 + 61 = 100$$
$$x_1 = 39$$

So we have one solution, $(x_1, x_2) = (39, 61)$.

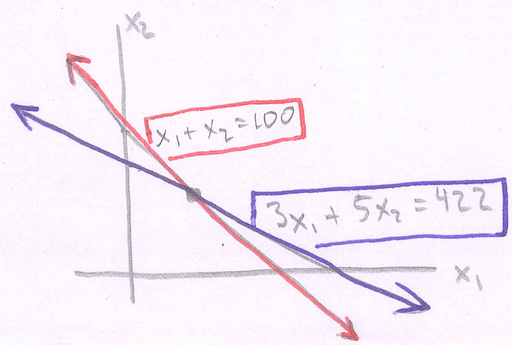
That's an algebraic way to get to the result, but there is also a geometric way, which is enlightening in its own way. Notice that $x_1 + x_2 = 100$ is a line:



where every point on the line represents a possible solution. Similarly, $3x_1 + 5x_2 = 422$ is also a line:



So, we are being asked to find solutions that lie on both lines, since each line represents solutions to its respective equation. Then, notice



the two lines intersect at precisely one point, which is why there is one solution to this system of equations.

QUESTION Can you ever have 2 solutions? No solutions? 10 solutions? Infinitely many solutions?

EXAMPLE 2

$$\begin{cases} ① & 6x_1 - 10x_2 = 0 \\ ② & -3x_1 + 5x_2 = 8 \end{cases}$$

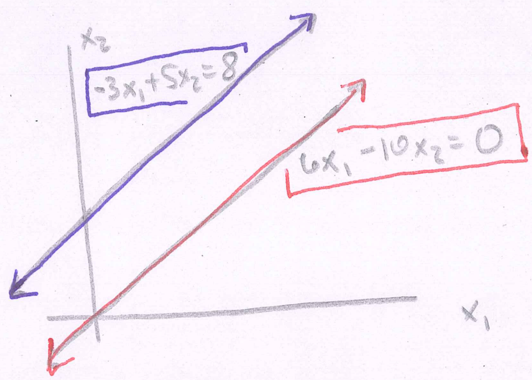
Let's eliminate the x_1 variable and solve for x_2.

$$\text{①} + 2\cdot\text{②}: \quad \begin{array}{r} 6x_1 - 10x_2 = 0 \\ + \ \underline{(-6x_1 + 10x_2 = 16)} \\ 0 = 16 \end{array}$$

... (?). 0 is clearly not equal to 16, so what's wrong?

It looks like it might be impossible to solve for x_2.

Consider the geometric picture:



The lines are parallel! (You can see this if you look back at the original equations and solve for the slope of each line.)

This means no solution for ① can ever be a solution to ②, and vice-versa. Thus, the system has <u>NO SOLUTION</u>.

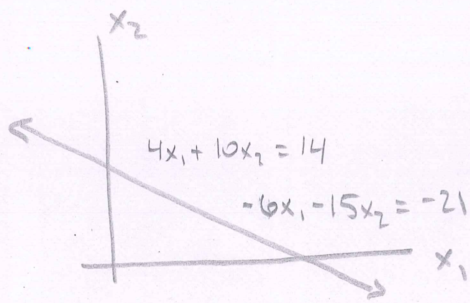
EXAMPLE 3

$$\begin{cases} \text{①} & 4x_1 + 10x_2 = 14 \\ \text{②} & -6x_1 - 15x_2 = -21 \end{cases}$$

Let's eliminate the x_1 variable and solve for x_2.

$$① + \tfrac{2}{3}\cdot②: \qquad \begin{array}{r} 4x_1 + 10x_2 = 14 \\ + \;(-4x_1 + {}^-10x_2 = {}^-14) \\ \hline 0 = 0 \end{array}$$

... (?). That is definitely true, so it seems like any value of x_2 could work ... Let's switch to the geometric picture.



The lines are the same! i.e. any solution to ① is also a solution to ②.

This is how we will handle this situation. We have a $\boxed{\text{"}\underline{\text{free}}\ \underline{\text{parameter}}\text{"}}$. Let $x_2 = s$ (s is the free parameter; i.e. x_2 can be anything) and use that to solve for x_1.

$$4x_1 + 10s = 14$$

$$x_1 = \frac{-5}{2}s + \frac{7}{2}$$

Then we say that all solutions are of the form

$$\begin{cases} x_1 = \frac{-5}{2}s + \frac{7}{2} \\ x_2 = s \end{cases}$$

for s being any real number.

EXERCISE

The previous examples were all in "two dimensions," i.e. there were two variables x_1 and x_2, and the equations described lines. Now assume we are in "three dimensions," i.e. there are three variables x_1, x_2, and x_3. What are possible solutions to

$$\begin{cases} ① & a_{11}x_1 + a_{12}x_2 + a_{13}x_3 = b_1 \\ ② & a_{21}x_1 + a_{22}x_2 + a_{23}x_3 = b_2 \\ ③ & a_{31}x_1 + a_{32}x_2 + a_{33}x_3 = b_3 \end{cases},$$

where a_{ij}, b_i are constants? Can you justify this geometrically? [HINT: $ax+by+cz=d$ is a plane ... .]

As the previous exercise hints at, there is nothing particularly special about linear equations of two variable x_1 and x_2.

One of our goals in this class is to understand how to solve systems of equations of n-variables, and the procedure we will use will motivate the study and use of matrices, which we will come to see as "concrete examples of linear transformations when we have a coordinate system."

Before we get there, though, let's define what we mean by a system of linear equations.

DEF A $\boxed{\text{linear equation}}$ is an equation of the form

$$a_1 x_1 + a_2 x_2 + a_3 x_3 + \cdots + a_n x_n = b,$$

where $a_1, a_2, a_3, \ldots, a_n$, and b are constants and $x_1, x_2, x_3, \ldots, x_n$ are variables.

DEF A $\boxed{\text{solution set}}$ is a set of all solutions to an equation.

Examples:

	solution set
linear equations in two variables:	lines
linear equations in three variables:	planes
linear equations in four or more variables:	hyperplanes

DEF A $\boxed{\text{system of linear equations}}$ is a collection of equations of the form

$$\begin{cases} a_{11} x_1 + a_{12} x_2 + \cdots + a_{1n} x_n = b_1 \\ a_{21} x_1 + a_{22} x_2 + \cdots + a_{2n} x_n = b_2 \\ \quad \vdots \\ a_{m1} x_1 + a_{m2} x_2 + \cdots + a_{mn} x_n = b_m \end{cases}$$

where a_{ij}, b_i are constants, m, n are integers, and $x_1, x_2, \ldots, x_n$ are variables.

In general, it can be kind of hard to solve an arbitrary system of linear equations, so we will start with special systems that are much easier to solve. Here's an example of an "easy" one:

EXAMPLE 4

Find all solutions to the system of linear equations:

$$\begin{aligned} x_1 - 2x_2 - 5x_3 + 3x_4 &= 2 \\ x_2 + 3x_3 - 4x_4 &= 7 \\ x_3 + 2x_4 &= -4 \\ x_4 &= 5 \end{aligned}$$

Clearly, $x_4 = 5$, so plug this into the third equation to solve for x_3!

$$\begin{aligned} x_3 + 2(5) &= -4 \\ x_3 &= -14 \end{aligned}$$

Now, $x_3 = -14$ and $x_4 = 5$. Plug this into the second equation

$$\begin{aligned} x_2 + 3(-14) - 4(5) &= 7 \\ x_2 &= 69 \end{aligned}$$

And finally, use the first equation to solve for $x_1 = 55$. So we see the solution is

$$(x_1, x_2, x_3, x_4) = (55, 69, -14, 5).$$

That system of equations certainly seems easier to solve than

say

$$\begin{aligned}
-2x_1 + 4x_2 + 11x_3 - 4x_4 &= 4 \\
3x_1 - 6x_2 - 15x_3 + 10x_4 &= 11 \\
2x_1 - 4x_2 - 10x_3 + 6x_4 &= 4 \\
-3x_1 + 7x_2 + 18x_3 - 13x_4 &= 1 .
\end{aligned}$$

Interestingly, though, this system of equations has the <u>same</u> <u>solution</u>.

In fact, you can turn this system into the system we were solving in the example.

<u>EXERCISE</u> Spend five minutes coming up with a methodology to turn this second system into the first (the one in the example). Just make this a thought experiment, no need to write it out.

Since the system in the example was so easy to solve, we should identify the underlying "form" and make a definition.

<u>DEF</u> A linear system is in <u>triangular form</u> (and we call it a <u>triangular system</u>) if it is of the form:

$$
\begin{cases}
a_{11} x_1 + a_{12} x_2 + a_{13} x_3 + \cdots + x_{1n} x_n = b_1 \\
\qquad a_{22} x_2 + a_{23} x_3 + \cdots + a_{2n} x_n = b_2 \\
\qquad\qquad a_{33} x_3 + \cdots + a_{3n} x_n = b_3 \\
\qquad\qquad\qquad \vdots \\
\qquad\qquad\qquad\qquad a_{nn} x_n = b_n
\end{cases}
$$

where $a_{11}, a_{22}, \ldots, a_{nn}$ are all non-zero constants.

These systems have some special properties (which make them easy to solve).

PROPERTIES

① Every variable is the leading variable of exactly one equation. By leading variable, we just mean the variable that shows up first (from left to right) when we order the variable in each equation ascending from left to right.

② There are the same number of equations as variables.

③ There is exactly one solution.

EXERCISE Work Example 6 in the text.

These triangular systems are idealized (in some sense, the best you can hope for). Let's consider a more general form now, that will allow for more possibilities (more solutions, different numbers of equations than variables ...). We'll start with an example.

EXAMPLE 6

Find all solutions to the system of linear equations

$$\begin{cases} 2x_1 - 4x_2 + 2x_3 + x_4 = 11 \\ \qquad\quad x_2 - x_3 + 2x_4 = 5 \\ \qquad\qquad\qquad\quad 3x_4 = 9 \end{cases}$$

It's almost triangular, so we can start at the bottom and work our way up. $3x_4 = 9 \implies x_4 = 3$.

Then plug into the second equation:

$$x_2 - x_3 + 2(3) = 5.$$

Ah! Two variables ... what is happening? Notice that in equation ①, we still can't solve for both x_2 and x_3 ...

What do we do?

The problem is simple. We have 3 equations, but 4 unknowns, meaning we have a $\boxed{\text{free variable.}}$ Notice that x_3 is not the leading term in any equation, so we can set

$$x_3 = s_1,$$ where s_1 is a parameter (any real number!)

and we will look for solutions to x_2 and x_1.

With $x_4 = 3$ and $x_3 = s_1$, plug into the second equation again:

$$x_2 - s_1 + 2(3) = 5$$

$$x_2 = -1 + s_1$$

And now, plug x_2, x_3, and x_4 into the first equation.

$$2x_1 - 4(\overbrace{-1+s_1}^{x_2}) + 2\overbrace{s_1}^{x_3} + \overbrace{3}^{x_4} = 11$$

$$2x_1 + 4 - 4s_1 + 2s_1 + 3 = 11$$

$$2x_1 = 4 + 2s_1$$

$$x_1 = 2 + s_1.$$

Then, our solution is:

$$\begin{cases} x_1 = 2 + s_1 \\ x_2 = -1 + s_1 \\ x_3 = s_1 \\ x_4 = 3 \end{cases}$$

Notice that we have an infinite number of solutions.

For any s_1 we pick, we get a solution, and since s_1 can be any real number, we get an infinite number of solutions.

QUESTION Geometrically, what does the solution set look like?

[HINT: there is only one parameter.]

The linear system of equations in the last example is not a triangular system, but it was relatively nice. We call systems of this form echelon systems.

DEF A linear system is in echelon form (and is called an echelon system) if the system is organized in a descending "stair step" pattern from left to right, so that the indices of the leading variable are strictly increasing from top to bottom. We allow an additional equation, with no variables, of the form $c = 0$, where c is not necessarily 0. If c is not 0, the system has no solution.

EXAMPLES OF ECHELON SYSTEMS

(1) EXAMPLE 6

(2)
$$\begin{cases} x_1 + 2x_2 - x_3 \quad\quad + 3x_5 = 7 \\ x_2 - 4x_3 \quad\quad + x_5 = -2 \\ x_4 - 2x_5 = 1. \end{cases}$$

(3)
$$\begin{cases} x_1 + 2x_2 - 5x_3 = -4 \\ -2x_2 + 4x_3 = 8 \\ 0 = 3 \end{cases}$$

⇐ This one will have no solution! Notice, if you start with a general system and try to turn it into a triangular system or echelon system, you may get an equation like "0=3", which is why we allow this in echelon form.

(4)
$$\begin{cases} x_1 + 2x_2 - 5x_3 = -4 \\ -2x_2 + 4x_3 = 8 \\ 0 = 0 \end{cases}$$

PROPERTIES (of Echelon Systems)

1. Every variable is the leading variable of at most one equation
2. There are no solutions, exactly one solution or infinitely many solutions.

EXERCISE

Suppose that $f(x) = a_1 e^{2x} + a_2 e^{-3x}$ is a solution to a differential equation. (You do not need to know how to solve differential equations to do this exercise!) If we know $f(0) = 5$ and $f'(0) = -1$, what are the values of a_1 and a_2? $\left[a_1 = \frac{14}{5}, \ a_2 = \frac{11}{5} \right]$ Can you write a similar question with n-unknown constants $a_1, \ldots, a_n$? How much more information do you need to solve for the constants?