Math 308 Conceptual Problems #6
Chapter 6 (6.1-6.2)

(1) Google the word 'eigenfaces' and look at the Wikipedia page, which has some pictures. (They come from artificial intelligence research on computer vision).

Here, a vector $\mathbf{v}$ represents an image. Basically $\mathbf{v}$ is the list of RGB color values of each pixel in the image, so $\mathbf{v} \in \mathbb{R}^N$ for some very large N. An 'eigenface' is an eigenvector for a matrix related to 'image vectors'.

(This is not a homework problem – just a neat application of linear algebra that's outside the scope of Math 308.)

(2) (Practice showing that something is a subspace). Suppose λ is an eigenvalue for the matrix A. Consider the λ-eigenspace of A:

$$E_\lambda(A) = \{\mathbf{v} \in \mathbb{R}^n : A\mathbf{v} = \lambda\mathbf{v}\},$$

the set of all vectors $\mathbf{v}$ satisfying the equation $A\mathbf{v} = \lambda\mathbf{v}$. One reason why $E_\lambda(A)$ is a subspace is because it is the nullspace of $A - \lambda I$. Show that $E_\lambda(A)$ is a subspace by directly checking the three conditions needed to be a subspace.

(3) Let $A = \begin{bmatrix} 4 & 1 \\ 3 & 2 \end{bmatrix}$.

(a) Compute the eigenvalues and eigenvectors of A.

(b) If λ is an eigenvalue of A then there is a vector $\mathbf{v}$ such that $A\mathbf{v} = \lambda\mathbf{v}$. Using this equation, show that λ^2 is an eigenvalue of A^2. What is an eigenvalue of A^{-1}? Now compute all eigenvalues and eigenvectors of A^2 and A^{-1}.

(c) Find a matrix B that shares an eigenvector with A but has different eigenvalues.

(d) Find an invertible matrix P and a diagonal matrix D so that $A = PDP^{-1}$. Then, compute A^{1000}.

(e) Suppose $\mathbf{v}$ is an eigenvector of an arbitrary matrix M, with eigenvalue λ. Show (using matrix algebra) that $\mathbf{v}$ is also an eigenvector of $M + I$, but with a different eigenvalue. What eigenvalue is it?

(4) (Reflections and projections)

(a) Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be the transformation from the conceptual problems for Chapter 4:

$$T(\mathbf{x}) = \frac{1}{3}\begin{bmatrix} -1 & -2 & 2 \\ -2 & 2 & 1 \\ 2 & 1 & 2 \end{bmatrix}\mathbf{x}.$$

Determine the eigenvalues of T, and find a basis for each eigenspace.

(b) Remember that T is supposed to be 'reflection across a plane S'. Explain what the eigenvalues and eigenvectors from (a) mean geometrically. What is their relationship to S? Why does it make sense for the eigenvalues to be 1 and -1?

(c) Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be the "averaging transformation":

$$T\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}\right) = \begin{bmatrix} \frac{1}{3}(x_1 + x_2 + x_3) \\ \frac{1}{3}(x_1 + x_2 + x_3) \\ \frac{1}{3}(x_1 + x_2 + x_3) \end{bmatrix}.$$

Find all eigenvalues and eigenspaces for T. Explain your answer (what does it mean in terms of 'averaging'?)

(5) (Rotations)

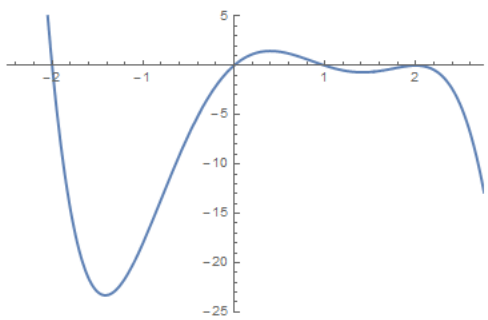
(a) Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be rotation by $\pi/3$. Compute the characteristic polynomial of T, and find any eigenvalues and eigenvectors.

(b) Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be a rotation in $\mathbb{R}^3$ by $\pi/3$ around some chosen axis L, a line through the origin in $\mathbb{R}^3$. **Without computing any matrices**, explain why $\lambda = 1$ is always an eigenvalue of T. What is the corresponding eigenspace?

(6) Find a 3×3 matrix A with eigenvectors $\mathbf{v}_1 = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ with $\lambda = 1$, $\mathbf{v}_2 = \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}$ with $\lambda = 2$ and $\mathbf{v}_3 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ with $\lambda = 10$.

(**Hint**: A must be diagonalizable, $A = PDP^{-1}$. Figure out P and D, then compute A directly.)

(7) Here is the graph of the characteristic polynomial of a 5×5 matrix A:



You may say "not enough information" for any of the following. Justify your answers.

(a) What polynomial has the above graph?
(b) What are the eigenvalues of A?
(c) Compute nullity$(A + 2I)$.
(d) What can you say about the value of rank$(A - 2I)$?
(e) Compute $\det(A - \lambda I)$.
(f) What can you say about $\det(A + I)$?
(g) Do the columns of A span $\mathbb{R}^5$?
(h) Is A diagonalizable?

(8) Suppose $T : \mathbb{R}^4 \to \mathbb{R}^4$ with $T(\mathbf{x}) = \mathbf{Ax}$ is a linear transformation such that

- $(0, 0, 1, 0)$ and $(0, 0, 0, 1)$ lie in the kernel of T, and
- all vectors of the form $(x_1, x_2, 0, 0)$ are reflected about the line $2x_1 - x_2 = 0$.

(a) Compute all the eigenvalues of A and a basis of each eigenspace.

(b) Is A invertible? Explain.

(c) Is A diagonalizable? If yes, write down its diagonalization (you can leave it as a product of matrices). If no, why not?