Math 308 Conceptual Problems (Sections 6.1-6.2)

(0) Google the word 'eigenfaces' and look at the Wikipedia page, which has some pictures. (They come from artificial intelligence research on computer vision).

Here, a vector $\vec{\mathbf{v}}$ represents an image. Basically $\vec{\mathbf{v}}$ is the list of RGB color values of each pixel in the image, so $\vec{\mathbf{v}} \in \mathbb{R}^N$ for some very large N. An 'eigenface' is an eigenvector for a matrix related to 'image vectors'.

(This is not a math problem – just a neat application of linear algebra that's outside the scope of Math 308.)

(1) (Practice showing that something is a subspace). Suppose λ is an eigenvalue for the matrix A. Let S be the λ-eigenspace:

$$S = \{\vec{\mathbf{v}} \in \mathbb{R}^n : A\vec{\mathbf{v}} = \lambda\vec{\mathbf{v}}\},$$

the set of all vectors $\vec{\mathbf{v}}$ satisfying the equation $A\vec{\mathbf{v}} = \lambda\vec{\mathbf{v}}$. In class, we said (roughly) that S is a subspace because $S = \mathrm{null}(A - \lambda I)$. For this problem, instead show that S is a subspace by checking the three conditions on S.

(2) Let $A = \begin{bmatrix} 4 & 1 \\ 3 & 2 \end{bmatrix}$ and $B = \begin{bmatrix} -2 & 1 \\ 3 & 2 \end{bmatrix}$.

(a) Compare the eigenvalues and eigenvectors of A, A^2 and A^{-1}. Are they similar or different?

(b) Same questions as (a) but for A, B and AB.

(c) Find an invertible matrix P and a diagonal matrix D so that $A = PDP^{-1}$. Then, compute A^{1000}. (Hint: you can use the method from the bonus problem of Midterm #2.)

(c) Suppose $\vec{\mathbf{v}}$ is an eigenvector of an arbitrary matrix M, with eigenvalue λ. Show (using matrix algebra) that $\vec{\mathbf{v}}$ is also an eigenvector of $M + I$, but with a different eigenvalue. What eigenvalue is it?

(3) (Reflections and projections)

(a) Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be the transformation from Worksheet 4, problem 1:

$$T(\vec{\mathbf{x}}) = \tfrac{1}{3}\begin{bmatrix} -1 & -2 & 2 \\ -2 & 2 & 1 \\ 2 & 1 & 2 \end{bmatrix}\vec{\mathbf{x}}.$$

Determine the eigenvalues of T, and find a basis for each eigenspace.
Note: You should find that the eigenvalues of T are 1 and -1.

(b) Remember that T is supposed to be 'reflection across a plane S'. Explain what the eigenvalues and eigenvectors from (a) mean geometrically. What is their relationship to S? Why does it make sense for the eigenvalues to be $1, -1$?

(c) Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be the "averaging transformation":

$$T\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}\right) = \begin{bmatrix} \frac{1}{3}(x_1 + x_2 + x_3) \\ \frac{1}{3}(x_1 + x_2 + x_3) \\ \frac{1}{3}(x_1 + x_2 + x_3) \end{bmatrix}.$$

Find all eigenvalues and eigenspaces for T. Explain your answer (what does it mean in terms of 'averaging'?)

(4) (Rotations)

(a) Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be rotation by $\pi/3$. Compute the characteristic polynomial of T, and find any eigenvalues and eigenvectors. (You can look up the matrix for T from previous worksheets or your notes from class).

(b) Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be a rotation in $\mathbb{R}^3$ by $\pi/3$ around some chosen axis L, a line through the origin in $\mathbb{R}^3$. **Without computing any matrices**, explain why $\lambda = 1$ is always an eigenvalue of T. What is the corresponding eigenspace?

(5) Find a 3×3 matrix A with eigenvectors $\vec{\mathbf{v}}_1 = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ with $\lambda = 1$, $\vec{\mathbf{v}}_2 = \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}$ with $\lambda = 2$ and $\vec{\mathbf{v}}_3 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ with $\lambda = 10$. You may take for granted that $\{\vec{\mathbf{v}}_1, \vec{\mathbf{v}}_2, \vec{\mathbf{v}}_3\}$ is a basis for $\mathbb{R}^3$.

(**Hint**: A must be diagonalizable, $A = PDP^{-1}$. Figure out P and D, then compute A directly.)