# Math 308 Conceptual Problems #5
## Chapter 5 (5.1-5.2)

(1) Find the determinant of the matrix

$$\begin{bmatrix} 1 & 3 & 2 \\ 0 & 1 & 1 \\ -2 & 0 & 4 \end{bmatrix}^3 \begin{bmatrix} 8 & 0 & 3 \\ -1 & 1 & 1 \\ 0 & 2 & 4 \end{bmatrix}^{-1}$$

(2) Compute $\det(5A^3)$ for the matrix $A$ below.

$$A = \begin{bmatrix} 0 & 2 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 3 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 4 & 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 5 & 6 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 6 & 5 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 4 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 3 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \end{bmatrix}$$

(3) This exercise uses the notion of *cofactors* of a matrix defined in Section 5.1. Recall that if $A$ is an $n \times n$ matrix, then $M_{ij}$ denotes the matrix obtained from $A$ by deleting the $i$th row and $j$th column of $A$.

Let $A = \begin{bmatrix} -2 & 0 & 2 \\ 1 & 1 & 1 \\ 3 & -1 & 5 \end{bmatrix}$.

(a) Compute all nine cofactors of $A$, as well as $\det(A)$. Let $B$ be the $3 \times 3$ matrix containing the cofactors, with each entry multiplied by the appropriate $\pm$ sign. So the $ij$-entry of $B$ is $(-1)^{i+j}\det(M_{ij})$.

(b) Compute $AB^T$. You should get a diagonal matrix with the same number in every diagonal entry, in other words, a multiple of the identity matrix. What multiple is it (in terms of $A$)?

(c) Fill in the blank (with a scalar) to make this equation true:

$$AB^T = (\ ?\ ) \cdot I, \quad \text{therefore} \quad A^{-1} = \frac{1}{(\ ?\ )} \cdot B^T.$$

(d) Compute the matrix $B^\top$ when $A$ is the $2 \times 2$ matrix

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix},$$

and then use it to compute a formula for $A^{-1}$. Does this agree with the formula for $A^{-1}$ given in the textbook?

The formula described above works for computing the inverse of any $n \times n$ matrix. This formula does not provide an *efficient algorithm (computer science* *terminology)* for computing inverses. Why? Compare it to the method of computing inverses using Gaussian elimination, and think about how many additions, subtractions, multiplications and divisions go into the two methods. The cofactor formula given above is however important and useful for mathematical arguments.

(4) Consider the plane $P$ in $\mathbb{R}^3$ spanned by the vectors $(1, 0, 1)$ and $(2, -1, 0)$.
  (a) Express $P$ as the solution set of a linear system in the variables $x, y, z$.
  (b) Compute the following determinant:
$$\det \begin{bmatrix} x & 1 & 2 \\ y & 0 & -1 \\ z & 1 & 0 \end{bmatrix}.$$
  (c) Check that the equation obtained by setting the determinant you just computed to zero has the same solutions as the linear system from (1). Why does this happen?
  (d) Can you now explain the formula for computing the cross product of two vectors in $\mathbb{R}^3$?
  (e) Generalize the process described in this example to $\mathbb{R}^4$. Take 3 linearly independent vectors (of your choice) and express the plane they span as the set of solutions to a linear equation by computing a determinant as above.

(5) (Determinants and Geometry)
  (a) Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be counter clockwise rotation by $\pi/3$, i.e. $T(\mathbf{x})$ is the vector obtained by rotating $\mathbf{x}$ counter clockwise by $\pi/3$ around $\mathbf{0}$. Without computing any matrices, what would you expect $\det(T)$ to be? (Does $T$ make areas larger or smaller?)
  Now check your answer by using the fact that the matrix for counter clockwise rotation by $\theta$ is
$$A = \begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix}.$$
  (b) Same question as (a), only this time let $T$ be the transformation that reflects $\mathbb{R}^2$ over the line $y = x$. That is, $T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} y \\ x \end{bmatrix}$.
  Guess what $\det(T)$ should be, then check by finding the matrix for $T$ and computing its determinant.
  (c) Rotation matrices in $\mathbb{R}^3$ are more complicated than in $\mathbb{R}^2$ because you have to specify an axis of rotation, which could be any line through the origin. Nonetheless, what would you expect $\det(T)$ to be?
  Look up the "basic 3D rotation matrices" on Wikipedia (`https://en.wikipedia.org/wiki/Rotation_matrix#In_three_dimensions`) and compute $\det(A)$ for each one.
  (d) Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be projection onto the $xy$-plane, so $T\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \begin{bmatrix} x \\ y \\ 0 \end{bmatrix}$. What is $\det(T)$? Guess first and then check using a matrix.

(6) (Determinants and Interpolation)
Suppose we want to make a quadratic polynomial

$$y = f(x) = a_0 + a_1 x + a_2 x^2$$

that passes through three specified points $\mathbf{p}_1 = (p_1, q_1)$, $\mathbf{p}_2 = (p_2, q_2), \mathbf{p}_3 = (p_3, q_3)$. Consider the equation

$$0 = \det \begin{bmatrix} 1 & x & x^2 & y \\ 1 & p_1 & p_1^2 & q_1 \\ 1 & p_2 & p_2^2 & q_2 \\ 1 & p_3 & p_3^2 & q_3 \end{bmatrix}.$$

The determinant above implicitly gives an equation $y = f(x)$ (it's easy to solve for $y$ since no $y^2, y^3$, etc terms appear).

(a) Write out the matrix above, using $\mathbf{p}_1 = (0,0), \mathbf{p}_2 = (1,1), \mathbf{p}_3 = (3,5)$ for the constants $p_i, q_i$, but leaving $x$ and $y$ as variables.
Solve the equation $\det(A) = 0$ to get $y =$ a quadratic polynomial in $x$. Check directly that the parabola passes through $\mathbf{p}_1, \mathbf{p}_2$, and $\mathbf{p}_3$.

(b) Why does part (a) succeed? Examine the matrix $A$ from part (a). If you plug in $(x, y) = \mathbf{p}_1 = (0, 0)$ to the first row of $A$, the first two rows will become the same. So, by the 'repeated rows' rule, the equation $\det(A) = 0$ must be true for those specific $x, y$ values. What does this mean about the polynomial $y = f(x)$?

What about if you plug in $(x, y) = (1, 1)$ or $(3, 5)$? Why (in terms of determinants) must the equation $y = f(x)$ be satisfied by these points?

(c) How could you use a determinant to make a cubic polynomial that passes through 4 given points? (It should require a $5 \times 5$ determinant.)

(7) Let $A(t) = b_1 \cos(\omega t) + b_2 \sin(\omega t)$ be the ambient temperature in your kitchen, which varies sinusoidally as function of time $t$. Assume $A(t)$ is known — that is, the values of $b_1$, $b_2$, and $\omega$ have been measured.

You have just cooked a cup of noodles. *Newton's Law of Cooling* states that the temperature function $y(t)$ of the noodles satisfies

$$y' = -k(y - A(t))$$

with $k$ a (known) positive constant. The "steady-state" solution (which $y$ approaches in the limit as $t$ increases) is of the form

$$y(t) = c_1 \cos(\omega t) + c_2 \sin(\omega t).$$

Use Newton's Law of Cooling to write $c_1$ and $c_2$ in terms of $b_1, b_2$. Write the linear relations you found as a system of equations with $kb_i$ on the right side. Then use an inverse matrix to find a formula for the steady-state solution.

(8) Let

$$f(t) = x_1 \sin(t) + x_2 \cos(t) + x_3 t \sin(t) + x_4 t \cos(t)$$

and

$$f'(t) = y_1 \sin(t) + y_2 \cos(t) + y_3 t \sin(t) + y_4 t \cos(t),$$

where the $x_i$ and $y_i$ are coefficients. Find a matrix $A$ expressing the vector $\mathbf{y} = (y_1, y_2, y_3, y_4)$ as $\mathbf{y} = A\mathbf{x}$ where $\mathbf{x} = (x_1, x_2, x_3, x_4)$. What is the interpretation of $A^2\mathbf{x}$ and $A^3\mathbf{x}$?

(9) Let $A = \begin{bmatrix} -4 & 3 \\ -1 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 0 & 1 \\ -1 & 3 \end{bmatrix}$.

(a) Compute $\det(A)$ and $\det(B)$.
(b) Using facts about determinants and without computing $AB$, argue that the columns of $AB$ form a basis of $\mathbb{R}^2$.
(c) Compute $A^{-1}$ and $B^{-1}$.
(d) Compute the coordinates of the point $(1, 1)$ in the basis formed by the columns of $AB$. Express your computation abstractly in terms of $A$ and $B$ before you use the specific $A$ and $B$ given above.
(e) What is the linear transformation $T$ that will undo the change of basis in (c) bringing your answer in (c) back to $(1, 1)$? Write $T$ with domain, codomain and matrix. Express your computation abstractly in terms of $A$ and $B$ before you use the specific $A$ and $B$ given above. Check that it works.