Math 308 Conceptual Problems #4
Chapter 4 (after 4.3)

(1) (after 4.1) Let S be a plane in $\mathbf{R}^3$ passing through the origin, so that S is a two-dimensional subspace of $\mathbf{R}^3$. Say that a linear transformation $T : \mathbf{R}^3 \to \mathbf{R}^3$ is a *reflection about* S if $T(\mathbf{v}) = \mathbf{v}$ for any vector $\mathbf{v}$ in S and $T(\mathbf{n}) = -\mathbf{n}$ whenever $\mathbf{n}$ is perpendicular to S. Let T be the linear transformation given by $T(\mathbf{x}) = A\mathbf{x}$, where A is the matrix

$$\frac{1}{3}\begin{bmatrix} -1 & -2 & 2 \\ -2 & 2 & 1 \\ 2 & 1 & 2 \end{bmatrix}.$$

This linear transformation is the reflection about a plane S. Find a basis for S.

(2) (after 4.2) (Geometry Question) In this problem we continue the Geoemtry Problem from Chapter 2, but now we work in $\mathbf{R}^3$. Consider the infinite linear system given by the equations $ax + by = 0$ where you should think of these having a z variable with zero coefficient.

(a) Describe the solution space of the above system.
(b) How many linearly independent solutions are there in this solution space? (i.e., what is the dimension of this solution space?)
(c) Write down a basis of the solution space.
(d) Express this solution space as the kernel of a finite matrix. What is the smallest size matrix that will do the job?
(e) If we keep on doing this example in higher and higher dimensional space, what happens to the dimension of the solution space?

(3) (after 4.2) Expand the set $\begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ -1 \\ 0 \\ 1 \end{bmatrix}$ to be a basis for the subspace $w + x + y + z = 0$.

(4) (after 4.3) Find an *invertible* $n \times n$ matrix A and an $n \times n$ matrix B such that $\operatorname{rank}(AB) \neq \operatorname{rank}(BA)$, or explain why such matrices cannot exist.

(5) (after 4.3) Find a 3×4 matrix A with nullity 2 and with

$$\operatorname{col}(A) = \operatorname{span}\left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 4 \\ -3 \\ 7 \end{bmatrix}, \begin{bmatrix} 3 \\ -2 \\ 5 \end{bmatrix} \right\},$$

or explain why such a matrix can't exist.

(6) (after 4.3) Find a 3×3 matrix A and a 3×3 matrix B, each with nullity 1, such that AB is the 0 matrix, or explain why such matrices cannot exist.