Math 308 Conceptual Problems #4
Chapter 4 (after 4.3)

(1) Let S be a plane in $\mathbb{R}^3$ passing through the origin, so that S is a two-dimensional subspace of $\mathbb{R}^3$. Say that a linear transformation $T : \mathbb{R}^3 \to \mathbb{R}^3$ is a *reflection about* S if $T(\mathbf{v}) = \mathbf{v}$ for any vector $\mathbf{v}$ in S and $T(\mathbf{n}) = -\mathbf{n}$ whenever $\mathbf{n}$ is perpendicular to S. Let T be the linear transformation given by $T(\mathbf{x}) = A\mathbf{x}$, where A is the matrix

$$\frac{1}{3}\begin{bmatrix} -1 & -2 & 2 \\ -2 & 2 & 1 \\ 2 & 1 & 2 \end{bmatrix}.$$

This linear transformation is the reflection about a plane S. Find a basis for S.

(2) Define $T : \mathbb{R}^3 \to \mathbb{R}^3$ by

$$T\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \begin{bmatrix} 0 & 3 & -2 \\ 1 & -2 & 2 \\ 2 & -6 & 5 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix}.$$

Let $\mathcal{V}$ be the set of all vectors that are fixed by T, which means that $\mathcal{V} = \{\mathbf{v} \in \mathbb{R}^3 \ : \ T(\mathbf{v}) = \mathbf{v}\}$.

(a) Show, using the definition of subspace, that $\mathcal{V}$ is a subspace of $\mathbb{R}^3$.

(b) Come up with an equation that also defines $\mathcal{V}$. (In other words, find a linear equation $ax + by + cz = d$ such that $\begin{bmatrix} x \\ y \\ z \end{bmatrix} \in \mathcal{V} \Leftrightarrow ax + by + cz = d$.)

(c) Geometrically, what kind of object is $\mathcal{V}$ (point/line/plane etc)?

(d) Find a basis for $\mathcal{V}$.

(3) We now build on Problem 6 from the conceptual problems from Chapter 2, but work in $\mathbb{R}^3$. Consider the infinite linear system of equations in three variables consisting of the equations $ax + by + 0z = 0$ as (a, b) moves along the unit circle in $\mathbb{R}^2$.

(a) Describe the solution space of the above system.
(b) How many linearly independent solutions are there in this solution space? (i.e., what is the dimension of this solution space?)
(c) Write down a basis of the solution space.
(d) Express this solution space as the kernel of a $n \times 3$ matrix. What is the smallest n that will do the job?
(e) If we keep on doing this example in higher and higher dimensional space by taking the system of equations $ax + by = 0$ as (a, b) moves along the unit circle, what happens to the dimension of the solution space? State your answer in dimensions 4 and 5 and then decide what happens in dimension n.

(4) Let V be the subspace of $\mathbb{R}^4$ defined as

$$V = \{(w, x, y, z) \in \mathbb{R}^4 \, : \, w + x + y + z = 0\}.$$

Check that the vectors $\begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ -1 \\ 0 \\ 1 \end{bmatrix}$ can be part of a basis for V. Then expand the set consisting of these two vectors to a basis of V.

(5) Find a 3×4 matrix A with nullity 2 and with column space

$$\mathrm{col}(A) = \mathrm{span}\left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 4 \\ -3 \\ 7 \end{bmatrix}, \begin{bmatrix} 3 \\ -2 \\ 5 \end{bmatrix} \right\},$$

or explain why such a matrix cannot exist.

(6) Let L in $\mathbb{R}^3$ be the line through the origin spanned by the vector $\mathbf{v} = \begin{bmatrix} 1 \\ 1 \\ 3 \end{bmatrix}$. Find the linear equations that define L, i.e., find a system of linear equations whose solutions are the points in L.

(7) Give an example of a linear transformation from $T : \mathbb{R}^2 \to \mathbb{R}^3$ with the following two properties:
(a) T is not one-to-one, and
(b)

$$\mathrm{range}(T) = \left\{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} \in \mathbb{R}^3 \, : \, x - y + 2z = 0 \right\};$$

or explain why this is not possible. If you give an example, you must include an explanation for why your linear transformation has the desired properties.

(8) Consider the following row equivalent matrices:

$$A = \begin{bmatrix} \mathbf{a}_1 & \mathbf{a}_2 & \mathbf{a}_3 & \mathbf{a}_4 \end{bmatrix} \sim \begin{bmatrix} 3 & 0 & -1 & 5 \\ 0 & 0 & 2 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix} = B = \begin{bmatrix} \mathbf{b}_1 & \mathbf{b}_2 & \mathbf{b}_3 & \mathbf{b}_4 \end{bmatrix}$$

Let $\mathrm{col}(A)$ be the column space of A. Answer the following with reasons.
(a) Is $\{\mathbf{a}_1, \mathbf{a}_3\}$ a basis for $\mathrm{col}(A)$?
(b) Is $\{\mathbf{a}_1 + \mathbf{a}_2, \mathbf{a}_3 + \mathbf{a}_4\}$ a basis for $\mathrm{col}(A)$?
(c) Is $\{\mathbf{a}_1 - \frac{1}{3}\mathbf{a}_3, \mathbf{a}_4\}$ a basis for $\mathrm{col}(A)$?
(d) Is $\{\mathbf{a}_1 + \mathbf{a}_3, \mathbf{a}_4\}$ a basis for $\mathrm{col}(A)$?

(9) Find, if possible, an example of a matrix A such that:

(a) A is 2×3, $\operatorname{col}(A) = \mathbb{R}^2$, and $\operatorname{null}(A) = \operatorname{span}\left(\begin{bmatrix}1\\1\\1\end{bmatrix}\right)$.

(b) A is 2×3, $\operatorname{col}(A) = \mathbb{R}^2$, and $\operatorname{null}(A) = \operatorname{span}\left(\begin{bmatrix}1\\-1\\0\end{bmatrix}, \begin{bmatrix}0\\1\\-1\end{bmatrix}\right)$.

(c) A is 2×2, $\operatorname{row}(A) = \operatorname{span}\left(\begin{bmatrix}1\\1\end{bmatrix}\right)$, and $\operatorname{null}(A) = \operatorname{span}\left(\begin{bmatrix}3\\-3\end{bmatrix}\right)$.

(d) A is 3×2, $\operatorname{row}(A) = \operatorname{span}\left(\begin{bmatrix}1\\1\end{bmatrix}\right)$, and $\operatorname{null}(A) = \operatorname{span}\left(\begin{bmatrix}3\\-3\end{bmatrix}\right)$.

(e) A is 2×2, $\operatorname{row}(A) = \operatorname{span}\left(\begin{bmatrix}1\\2\end{bmatrix}\right)$, and $\operatorname{null}(A) = \operatorname{span}\left(\begin{bmatrix}3\\1\end{bmatrix}\right)$.

(f) A is 2×3, $\operatorname{col}(A) = \operatorname{span}\left(\begin{bmatrix}1\\2\end{bmatrix}\right)$, and $\operatorname{null}(A) = \operatorname{span}\left(\begin{bmatrix}1\\1\\1\end{bmatrix}\right)$.

(10) Let $B = \begin{bmatrix}1 & 0 & 0\\0 & 1 & 0\\0 & 0 & 0\end{bmatrix}$. Note that $\operatorname{null}(B) = \operatorname{span}\left(\begin{bmatrix}0\\0\\1\end{bmatrix}\right)$.

(a) Find if possible, a 3×3 matrix A, where BA has nullity 2. If you find an example, what is the nullity of the matrix A that you found? Can you find an example with a different nullity?

(b) Find if possible, a 3×3 matrix A, where BA has nullity 1. If you find an example, what is the nullity of the matrix A that you found? Can you find an example with a different nullity?

(c) Find if possible, a 3×3 matrix A, where BA has nullity 0. If you find an example, what is the nullity of the matrix A that you found? Can you find an example with a different nullity?