Math 308 Conceptual Problems #3
Chapter 3 (after 3.3)

(1) (after 3.1)
   (a) Find a linear transformation $T : \mathbb{R}^2 \to \mathbb{R}^3$ such that

$$T\left(\begin{bmatrix}1\\1\end{bmatrix}\right) = \begin{bmatrix}1\\0\\2\end{bmatrix}, \quad T\left(\begin{bmatrix}0\\2\end{bmatrix}\right) = \begin{bmatrix}-2\\2\\0\end{bmatrix}, \quad \text{and } T\left(\begin{bmatrix}1\\3\end{bmatrix}\right) = \begin{bmatrix}-1\\2\\2\end{bmatrix},$$
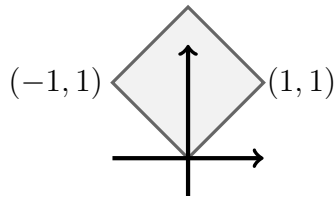
   or if it's impossible, explain why.
   (b) How does your answer change if the third condition changes to

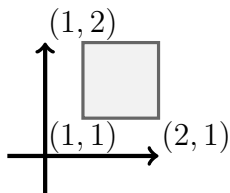$$T\left(\begin{bmatrix}1\\3\end{bmatrix}\right) = \begin{bmatrix}-1\\2\\1\end{bmatrix}?$$

(2) (after 3.1) For each of the shaded regions below, find a linear transformation $T : \mathbb{R}^2 \to \mathbb{R}^2$ that takes the unit square

$$\mathcal{U} = \{(x, y) \in \mathbb{R}^2 : 0 \le x \le 1 \text{ and } 0 \le y \le 1\}$$

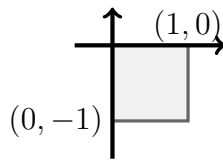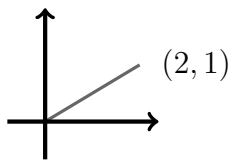   to the given region, or explain why it's impossible to do so.

(a)

$(-1, 1)$ $(1, 1)$

(b)

$(1, 2)$

$(1, 1)$ $(2, 1)$

(c)

$(1, 0)$

$(0, -1)$

(d)

$(2, 1)$

(3) (after 3.1) Assume $T : \mathbb{R}^m \to \mathbb{R}^n$ is a linear transformation.
   (a) Suppose there is a nonzero vector $\mathbf{x} \in \mathbb{R}^m$ such that $T(\mathbf{x}) = \mathbf{0}$. Is it possible that T is one-to-one? Give an example, or explain why it's not possible.
   (b) Suppose there is a nonzero vector $\mathbf{x} \in \mathbb{R}^m$ such that $T(\mathbf{x}) = \mathbf{0}$. Is it possible that T is onto? Give an example, or explain why it's not possible.
   (c) Suppose that $\mathbf{u}$ and $\mathbf{v}$ are linearly dependent vectors in $\mathbb{R}^m$. Show that $T(\mathbf{u})$ and $T(\mathbf{v})$ are also linearly dependent.

(d) Suppose that $\mathbf{u}$ and $\mathbf{v}$ are linearly *independent* vectors in $\mathbb{R}^m$. Is it guaranteed that $T(\mathbf{u})$ and $T(\mathbf{v})$ are also linearly independent? If yes, explain why. If no, give an example where this is not the case.

(4) (after 3.1) Let $\mathbf{w} \in \mathbb{R}^n$ and suppose that $T : \mathbb{R}^n \to \mathbb{R}^n$ is given by $T(\mathbf{v}) = \mathbf{v} + \mathbf{w}$. Determine the exact conditions on $\mathbf{w}$ that make T a linear transformation. (First, show that if your condition on $\mathbf{w}$ is satisfied, then T is a linear transformation. Then show that if your condition on $\mathbf{w}$ is *not* satisfied, then T is *not* a linear transformation.)

(5) (Geometry Question, after 3.1) Suppose we are given the unit square A in the plane with corners $(0, 0)$, $(1, 0)$, $(1, 1)$ and $(0, 1)$.

(a) Find a linear transformation T that sends A to the parallelogram B with corners $(0, 0)$, $(1, 2)$, $(2, 2)$ and $(1, 0)$.
(b) Is the linear transformation T unique? Why or why not?
(c) What linear transformation S, other than the identity, would send A to itself?
(d) Is there a linear transformation whose overall effect would be to send A to B and then translate B in the horizontal direction by one unit? Explain.
(e) Find a linear transformation that will send A to a parallelogram C of area 4 while still keeping $(0, 0)$ and $(1, 0)$ as two of its corners.
(f) What is the general formula for the linear transformation that sends A to a parallelogram of area k while still keeping $(0, 0)$ and $(1, 0)$ as two of its corners?

(6) (Geometry Question, after 3.1) Consider the triangle Δ in $\mathbb{R}^3$ with corners $(1, 0, 0)$, $(0, 1, 0)$, $(0, 0, 1)$.
(a) Find the image of Δ under the projection that sends $(x, y, z) \in \mathbb{R}^3 \mapsto (x, y) \in \mathbb{R}^2$. Is this a linear transformation?
(b) What the area of the image of Δ after the projection?
(c) Is there a linear transformation that will send Δ to the xy-plane so that the image has the same area as Δ, and one of the corners of the image is $(0, 0)$?

(7) (after 3.1) Say we have linear transformations $T : \mathbb{R}^3 \to \mathbb{R}^2$ and $S : \mathbb{R}^2 \to \mathbb{R}^4$. Let $S \circ T : \mathbb{R}^3 \to \mathbb{R}^4$ be the composition (that is, $\mathbb{R}^3 \xrightarrow{T} \mathbb{R}^2 \xrightarrow{S} \mathbb{R}^4$).
Can $S \circ T$ be one-to-one?
(Hint: Start by thinking about all $\mathbf{x}$ such that $T(\mathbf{x}) = \mathbf{0}$.)

(8) (after 3.2) Let
$$A = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 1 & 4 \end{bmatrix}.$$
Find a 3×2 matrix B with $AB = I_2$. Is there more than one matrix B with this property? Justify your answer.

(9) (after 3.2) Find a 2×2 matrix A, which is not the zero or identity matrix, satisfying each of the following equations.
a) $A^2 = 0$
b) $A^2 = A$
c) $A^2 = I_2$

(10) (after 3.2) Let
$$B = \begin{bmatrix} 1 & z \\ 4 & 3 \end{bmatrix}.$$
Find all values of z such that the linear transformation T induced by B fixes no line in $\mathbb{R}^2$. (By "fixing a line" we mean that $T(\mathbf{v}) = \mathbf{v}$ for every point $\mathbf{v}$ on the line.)

(11) (after 3.2) Suppose that A is a matrix and $\mathbf{b}$ is a vector in $\mathbb{R}^2$. Suppose further that
$$\mathbf{v} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \text{ and } \mathbf{w} = \begin{bmatrix} 5 \\ -3 \\ 1 \end{bmatrix} \text{ are both solutions to the equation } A\mathbf{x} = \mathbf{b}.$$
    (a) How many solutions does the equation $A\mathbf{x} = \mathbf{b}$ have? Explain your answer.

    (b) Find a nontrivial solution to the homogeneous system $A\mathbf{x} = \mathbf{0}$. Justify that it is indeed a solution.

(12) (after 3.3)

    (a) Find a linear transformation $T : \mathbb{R}^2 \to \mathbb{R}^2$ such that $T(\mathbf{x}) = A\mathbf{x}$ that reflects a vector (x_1, x_2) about the x_2-axis.

    (b) Find a linear transformation $S : \mathbb{R}^2 \to \mathbb{R}^2$ such that $T(\mathbf{x}) = B\mathbf{x}$ that rotates a vector (x_1, x_2) counterclockwise by 135 degrees.

    (c) Find a linear transformation (with domain and codomain) that has the effect of first reflecting as in (a) and then rotating as in (b). Give the matrix of this transformation explicitly. How is this transformation related to T and S?

    (d) Find the matrix representing the linear transformation that first rotates as in (b), then reflects as in (a), and then rotates backwards (i.e., clockwise by 135 degrees).

    (e) What matrix do you get if you repeat the sequence in part (d) ten times? Write this matrix in terms of A and B. Can you write this matrix explicitly?

(13) (after 3.3) Solve for the matrix X in the equation $AX(D + BX)^{-1} = C$. Assume that all matrices are $n \times n$ and invertible as needed.

(14) (after 3.3) Find all values of x for which the following matrix does not have an inverse.
$$\begin{bmatrix} 1 & 0 & 3 \\ -2 & x & 1 \\ 4 & -1 & 2 \end{bmatrix}$$