Math 308 Conceptual Problems #2
Chapter 2 (Sections 2.1-2.3)

(1) (after 2.1) When Jake works from home, he typically spends 40 minutes of each hour on research, and 10 on teaching, and drinks half a cup of coffee. (The remaining time is spent on the internet.) For each hour he works in the math department, he spends around 20 minutes on research and 30 on teaching, and doesn't drink any coffee. Lastly, if he works at a coffeeshop for an hour, he spends 25 minutes each on research and teaching, and drinks a cup of coffee.
(**Note:** be careful about units of minutes versus hours.)

(a) Last week, Jake spent 10 hours working from home, 15 hours working in his office in Padelford Hall, and 2 hours working at Cafe Allegro. Compute what was accomplished, and express the result as a vector equation.

(b) This week, Jake has 15 hours of research to work on and 10 hours of work related to teaching. He also wants 11 cups of coffee, because... of... very important reasons. How much time should he spend working from home, from his office, and from the coffeeshop?

(c) Describe the situation in part (b) as a vector equation and a matrix equation $A\mathbf{t} = \mathbf{w}$. What do the vectors $\mathbf{t}$ and $\mathbf{w}$ mean in this context? For which other vectors $\mathbf{w}$ does the equation $A\mathbf{t} = \mathbf{w}$ have a solution?

(d) Jake tries working in the math department lounge for an hour, and gets 30 minutes of research and 20 minutes of teaching work done, while having time to drink $\frac{1}{3}$ of a cup of coffee. Not bad. But Jake's colleague Vasu claims that there's no need to work in the lounge – the other options already give enough flexibility. Is he right? Explain mathematically.

(2) (after 2.1) Find a 3×4 matrix A, in *reduced* echelon form, with free variable x_3, such that the general solution of the equation $A\mathbf{x} = \begin{bmatrix} -1 \\ 1 \\ 6 \end{bmatrix}$ is

$$\mathbf{x} = \begin{bmatrix} -1 \\ 1 \\ 0 \\ 6 \end{bmatrix} + s \begin{bmatrix} -1 \\ 2 \\ 1 \\ 0 \end{bmatrix},$$

where s is any real number.

(3) (after 2.2) Find all values z_1 and z_2 such that $(2, -1, 3)$, $(1, 2, 2)$, and $(-4, z_1, z_2)$ do not span $\mathbb{R}^3$.

(4) (after 2.2)
(a) The set

$$P = \left\{ \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} : 2x_1 - x_2 + 4x_3 = 0 \right\}$$

is a plane in $\mathbb{R}^3$. Find two vectors $\mathbf{u}_1$, $\mathbf{u}_2 \in \mathbb{R}^3$ so that $\text{span}\{\mathbf{u}_1, \mathbf{u}_2\} = P$. Explain your answer.

(b) Consider the three vectors $\mathbf{u}_1 = \begin{bmatrix} 2 \\ 7 \\ -1 \end{bmatrix}$, $\mathbf{u}_2 = \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}$, $\mathbf{u}_3 = \begin{bmatrix} -5 \\ 8 \\ -5 \end{bmatrix}$. Let $\mathbf{b} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$
be an arbitrary vector in $\mathbb{R}^3$. Use Gaussian elimination to determine which
vectors $\mathbf{b}$ are in span$\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$. (**Hint:** You might look back at Problem 8 in
Conceptual Problems #1.)
Without further calculation, conclude that span$\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$ is a plane in $\mathbb{R}^3$
and identify an equation of the plane in the form $ax_1 + bx_2 + cx_3 = 0$.

(5) (after 2.3) Let $\mathbf{a_1} = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$, $\mathbf{a_2} = \begin{bmatrix} 0 \\ 2 \\ 3 \end{bmatrix}$, and $\mathbf{a_3} = \begin{bmatrix} t \\ -3 \\ -7 \end{bmatrix}$.

(a) Find all values of t for which there will be a unique solution to
$\mathbf{a_1}x_1 + \mathbf{a_2}x_2 + \mathbf{a_3}x_3 = \mathbf{b}$ for every vector $\mathbf{b}$ in $\mathbb{R}^3$. Explain your answer.
(b) Are the vectors $\mathbf{a_1}$ and $\mathbf{a_2}$ from part (a) linearly independent? Explain your
answer.

(c) Let $\mathbf{a_1}$, $\mathbf{a_2}$ and $\mathbf{a_3}$ be as in (a). Let $\mathbf{a_4} = \begin{bmatrix} 1 \\ 4 \\ -5 \end{bmatrix}$. Without doing any further
calculations, find all values of t for which there will be a unique solutions to
$\mathbf{a_1}y_1 + \mathbf{a_2}y_2 + \mathbf{a_3}y_3 + \mathbf{a_4}y_4 = \mathbf{c}$ for every vector $\mathbf{c}$ in $\mathbb{R}^3$. Explain your answer.

(6) (after 2.3) Consider the infinite system of linear equations in two variables given by
$ax + by = 0$ where (a, b) moves along the unit circle in the plane.

(a) How many solutions does this system have?
(b) What is the smallest number of equations in the above system that have the
same solution set? Write down two separate such linear systems, in vector
form.
(c) What happens to the infinite linear system if you add the equation $0x + 0y = 0$
to it?
(d) What happens to the infinite linear system if by accident one of the equations
was recorded as $ax + by = 0.00001$?

Explain all your answers in words.

(7) (after 2.3) For each of the situations described below, **give an example** (if it's
possible) or **explain why it's not possible**.
(a) A set of vectors that does not span $\mathbb{R}^3$. After adding one more vector, the set
does span $\mathbb{R}^3$.
(b) A set of vectors that are linearly dependent. After adding one more vector, the
set becomes linearly independent.
(c) A set of vectors in $\mathbb{R}^3$ with the following properties (four possibilities):

spans $\mathbb{R}^3$,	spans $\mathbb{R}^3$,
linearly independent	linearly dependent
doesn't span $\mathbb{R}^3$,	doesn't span $\mathbb{R}^3$,
linearly independent	linearly dependent

For each case that is *possible*, how many vectors could be in the set? (State
any constraints, as in "there must be at least..." or "at most...")
(e) A system of equations with a unique solution. After adding another equation
to the system, the new system has infinitely-many solutions.
(f) A system of equations without any solutions. After deleting an equation, the
system has infinitely-many solutions.

(8) (after 2.3) In each of the following cases, either find an example that contradicts the statement showing that it is false, or explain why the statement is always true.

(a) If $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$ is a spanning set for $\mathbb{R}^n$, then $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \mathbf{u}_4\}$ is also a spanning set for $\mathbb{R}^n$. What are all possible values of n for which three vectors $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3$ can span $\mathbb{R}^n$?

(b) If $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$ is a spanning set for $\mathbb{R}^n$, then $\{\mathbf{u}_1, \mathbf{u}_1 + \mathbf{u}_2, \mathbf{u}_1 - \mathbf{u}_3\}$ also spans $\mathbb{R}^n$.

(c) If $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3$ are linearly independent, then $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \mathbf{u}_4$ are also linearly independent.

(d) If $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3$ are linearly independent, then $\mathbf{u}_1, \mathbf{u}_1 + \mathbf{u}_2, \mathbf{u}_1 - \mathbf{u}_3$ are also linearly independent.

(e) If $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3$ do not span $\mathbb{R}^3$, then there is a plane P in $\mathbb{R}^3$ that contain all of them. (Bonus: how can we find this plane? Does the plane go through the origin?)

(9) (after 2.3) Recall that if we have m vectors $\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_m$ in $\mathbb{R}^n$, then we can form the matrix A whose columns are $\mathbf{u}_1, \ldots, \mathbf{u}_m$. Let B be the echelon form of A. All the questions below are based on such a matrix B. Most questions have a yes/no answer. Give full reasons for all answers.

Suppose we are given the following matrix B:

$$\begin{bmatrix} 3 & 0 & -1 & 5 \\ 0 & 0 & 2 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Note that the columns of B are not $\mathbf{u}_1, \ldots, \mathbf{u}_m$.

(a) What is n?
(b) What is m?
(c) Are $\mathbf{u}_1, \ldots, \mathbf{u}_m$ linearly independent?
(d) Does $\{\mathbf{u}_1, \ldots, \mathbf{u}_m\}$ span $\mathbb{R}^n$?
(e) Looking at B can you write down a subset of the original set $\{\mathbf{u}_1, \ldots, \mathbf{u}_m\}$ that would be guaranteed to be linearly independent?
(f) Is there a subset of the original set $\{\mathbf{u}_1, \ldots, \mathbf{u}_m\}$ that would be guaranteed to span $\mathbb{R}^n$?
(g) Write down a $\mathbf{b} \in \mathbb{R}^n$ for which $B\mathbf{x} = \mathbf{b}$ does not have a solution.
(h) Write down a $\mathbf{b} \in \mathbb{R}^n$ for which $B\mathbf{x} = \mathbf{b}$ has a solution.
(i) Write down a $\mathbf{b} \in \mathbb{R}^n$ for which $B\mathbf{x} = \mathbf{b}$ has a unique solution.
(j) Is there a new vector $\mathbf{w} \in \mathbb{R}^n$ that you could add to the set $\{\mathbf{u}_1, \ldots, \mathbf{u}_m\}$ to guarantee that $\{\mathbf{u}_1, \ldots, \mathbf{u}_m, \mathbf{w}\}$ will span $\mathbb{R}^n$?
(k) Is there a column of B that is in the span of the rest? If so, find it.
(l) Looking at B do you see a $\mathbf{u}_i$ that is in the span of the others? How can you identify it?
(m) Put B into reduced echelon form.
(n) Write down a non-zero solution of $A\mathbf{x} = \mathbf{0}$ if you can.
(o) How many free variables are there in the set of solutions to $A\mathbf{x} = \mathbf{b}$ when there is a solution?
(p) If you erased the last row of zeros in B then would the columns of the resulting matrix be linearly independent?
(q) Can you add rows to B to make the columns of the new matrix linearly independent? If yes, give an example of the new matrix you would construct.