Math 308 Conceptual Problems #2 (Sections 2.1-2.3)

(1) When Jake works from $\vec{h}$ome, he typically spends 40 minutes of each hour on research, and 10 on teaching, and drinks half a cup of coffee. (The remaining time is spent on the internet.) For each hour he works in the math $\vec{d}$epartment, he spends around 20 minutes on research and 30 on teaching, and doesn't drink any coffee. Lastly, if he works at a $\vec{c}$offeeshop for an hour, he spends 25 minutes each on research and teaching, and drinks a cup of coffee.

(**Note:** be careful about units of minutes versus hours.)

(a) Last week, Jake spent 10 hours working from home, 15 hours working in his office in Padelford Hall, and 2 hours working at Cafe Allegro. Compute what was accomplished, and express the result as a vector equation.

(b) This week, Jake has 15 hours of research to work on and 10 hours of work related to teaching. He also wants 11 cups of coffee, because... of... very important reasons. How much time should he spend working from home, from his office, and from the coffeeshop?

(c) Describe the situation in part (b) as a vector equation and a matrix equation $A\vec{t} = \vec{w}$. What do the vectors $\vec{t}$ and $\vec{w}$ mean in this context? For which other vectors $\vec{w}$ does the equation $A\vec{t} = \vec{w}$ have a solution?

(d) Jake tries working in the math department $\vec{l}$ounge for an hour, and gets 30 minutes of research and 20 minutes of teaching work done, while having time to drink $\frac{1}{3}$ of a cup of coffee. Not bad. But Jake's colleague Vasu claims that there's no need to work in the lounge – the other options already give enough flexibility. Is he right? Explain mathematically.

(2) (after 2.1) Find a 3×4 matrix A, in *reduced* echelon form, with free variable x_3, such that the general solution of the equation $A\mathbf{x} = \begin{bmatrix} -1 \\ 1 \\ 6 \end{bmatrix}$ is

$$\mathbf{x} = \begin{bmatrix} -1 \\ 1 \\ 0 \\ 6 \end{bmatrix} + s \begin{bmatrix} -1 \\ 2 \\ 1 \\ 0 \end{bmatrix},$$

where s is any real number.

(3) (after 2.2) Find all values z_1 and z_2 such that $(2, -1, 3)$, $(1, 2, 2)$, and $(-4, z_1, z_2)$ do not span $\mathbb{R}^3$.

(4) (after 2.3) (a) Let $\mathbf{a_1} = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$, $\mathbf{a_2} = \begin{bmatrix} 0 \\ 2 \\ 3 \end{bmatrix}$, and $\mathbf{a_3} = \begin{bmatrix} t \\ -3 \\ -7 \end{bmatrix}$. Find all values of t for which there will be a unique solution to $\mathbf{a_1}x_1 + \mathbf{a_2}x_2 + \mathbf{a_3}x_3 = \mathbf{b}$ for every vector $\mathbf{b}$ in $\mathbb{R}^3$. Explain your answer.

(b) Are the vectors $\mathbf{a_1}$ and $\mathbf{a_2}$ from part (a) linearly independent? Explain your answer.

(c) Let $\mathbf{a_1}$, $\mathbf{a_2}$ and $\mathbf{a_3}$ be as in (a). Let $\mathbf{a_4} = \begin{bmatrix} 1 \\ 4 \\ -5 \end{bmatrix}$. Without doing any further calculations, find all values of t for which there will be a unique solutions to $\mathbf{a_1}y_1 + \mathbf{a_2}y_2 + \mathbf{a_3}y_3 + \mathbf{a_4}y_4 = \mathbf{c}$ for every vector $\mathbf{c}$ in $\mathbb{R}^3$. Explain your answer.

(5) (after 2.3) (Geometry Question) Consider the infinite system of linear equations in two variables given by $ax + by = 0$ where (a, b) moves along the unit circle in the plane.

Recall that the vector (a, b) is the normal to the line with equation $ax + by = 0$, i.e., (a, b) is perpendicular to the line.

(a) How many solutions does this system have?
(b) How many linearly independent equations in the above system give you the same set of solutions? Write down two separate such linear systems, in vector form.
(c) What happens to the infinite linear system if you add to it the equation $0x + 0y = 0$?
(d) What happens to the infinite linear system if one of the equations slightly perturbs to $ax + by = c$ where c is a small positive number?
Explain all your answers in words.

(6) For each of the situations described below, **give an example** (if it's possible) or **explain why it's not possible**.

(a) A set of vectors that does not span $\mathbb{R}^3$. After adding one more vector, the set does span $\mathbb{R}^3$.
(b) A set of vectors that are linearly dependent. After adding one more vector, the set becomes linearly independent.
(c) A set of vectors in $\mathbb{R}^3$ with the following properties (four possibilities):

spans $\mathbb{R}^3$, linearly independent	spans $\mathbb{R}^3$, linearly dependent
doesn't span $\mathbb{R}^3$, linearly independent	doesn't span $\mathbb{R}^3$, linearly dependent

For each case that is *possible*, how many vectors could be in the set? (State any constraints, as in "there must be at least..." or "at most...")

(e) A system of equations with a unique solution. After adding another equation to the system, the new system has infinitely-many solutions.
(f) * A system of equations without any solutions. After deleting an equation, the system has infinitely-many solutions.

(7) Recall that if we have m vectors $u_1, u_2, \ldots, u_m$ in $\mathbb{R}^n$, then we can form the matrix A whose columns are $u_1, \ldots, u_m$. Let B be the echelon form of A. All the questions below are based on such a matrix B. Most questions have a yes/no answer, but I am mostly interested in your reasons for the answer. Give full reasons for all answers.

Suppose we are given the following matrix B:

$$\begin{pmatrix} 3 & 0 & -1 & 5 \\ 0 & 0 & 2 & -1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

(a) What is n?
(b) What is m?
(c) Are $u_1, \ldots, u_m$ linearly independent?
(d) Does $\{u_1, \ldots, u_m\}$ span $\mathbb{R}^n$?
(e) Looking at B can you write down a subset of the original set $\{u_1, \ldots, u_m\}$ that would be guaranteed to be linearly independent?
(f) Is there a subset of the original set $\{u_1, \ldots, u_m\}$ that would be guaranteed to span $\mathbb{R}^n$?
(g) Write down a $b \in \mathbb{R}^n$ for which $Bx = b$ does not have a solution.
(h) Write down a $b \in \mathbb{R}^n$ for which $Bx = b$ has a solution.
(i) Write down a $b \in \mathbb{R}^n$ for which $Bx = b$ has a unique solution.

(j) Is there a new vector $w \in \mathbb{R}^n$ that you could add to the set $\{u_1, \dots, u_m\}$ to guarantee that $\{u_1, \dots, u_m, w\}$ will span $\mathbb{R}^n$?
(k) Is there a column of B that is in the span of the rest? If so, find it.
(l) Looking at B do you see a u_i that is in the span of the others? How can you identify it?
(m) Assuming that no row of A was a zero row, how many planes are being used to cut out the solutions of $Ax = 0$?
(n) Looking at B, how many planes are needed at the minimum to cut out the solution set of $Ax = 0$?
(o) If your answers to the last two questions are different, why is there a difference?
(p) Put B into reduced echelon form.
(q) Write down a non-zero solution of $Ax = 0$ if you can.
(r) How many free variables are there in the set of solutions to $Ax = b$ when there is a solution?
(s) If you erased the last row of zeros in B then would the columns of the resulting matrix be linearly independent?
(t) Can you add rows to B to make the columns of the new matrix linearly independent? If yes, give an example of the new matrix you would construct.

(8) Recall that the neighborhood of a vertex in a graph consists of the vertex itself, together with all vertices that are connected to it by an edge. Each graph has a variable x_i associated with the i-th vertex, and the vertex has a known value that is equal to the sum of the variables for all neighborhood vertices.
  (a) Start with a graph with 5 vertices forming a pentagon, with edges joining vertices 1&2, 2&3, 3&4, 4&5, and 5&1. Then draw an edge joining vertices 2 and 4, and an edge joining vertices 2 and 5. The known values at vertices 1 through 5 are, respectively, 2, 1, -1, 3, and 5.
    (i) Find the augmented matrix for the system of equations satisfied by x_1, x_2, x_3, x_4, x_5.
    (ii) Using elementary row operations, find an equivalent reduced echelon matrix.
    (iii) Find the solution. Check that it satisfies the original equations.
  (b) Start with a graph with 4 vertices forming a square, and connect the opposite vertices 2&4. The known values at vertices 1 through 4 are, respectively, -2, 1, 5, 1. Same questions (i)–(iii) as in part (a).