Math 308 Conceptual Problems #1
Chapter 1 (Sections 1.1 and 1.2)

(1) (1.1: Modeling) Matt is a software engineer writing a script involving 6 tasks. Each must be done one after the other. Let t_i be the time needed to complete the ith task. These times have a certain structure:

- The total time needed to complete any 3 adjacent tasks is half the total time required to complete the next two tasks.
- The second task takes 1 second.
- The fourth task takes 10 seconds.

(a) Write an augmented matrix for the system of equations describing the length of each task.
(b) Reduce this augmented matrix to reduced echelon form.
(c) Suppose he knows additionally that the sixth task will take 20 seconds and the first three tasks together will take 50 seconds. Write the extra rows that you would add to your answer in (b) to take account of this new information.
(d) Solve the system of equations in (c).

(2) (1.1: Modeling) Before paying employee bonuses and state and federal taxes, a company earns profits of \$103,000. The company pays employees a bonus equal to 5% of after-tax profits. State tax is 5% of profits (after bonuses are paid). Finally, federal tax is 40% of profits (after bonuses and state tax are paid). Calculate the amounts paid in bonuses, state tax and federal tax.

(3) (1.1: Geometry) For each part below, give an example of a linear system of equations in two variables that has the given property. In each case, draw the lines corresponding to the solutions of the equations in the system.

(a) has no solution
(b) has exactly one solution
(c) has infinitely many solutions

(i) Add or remove equations in (b) to make an inconsistent system.
(ii) Add or remove equations in (b) to create infinitely many solutions.
(iii) Add or remove equations in (b) so that the solution space remains unchanged.
(iv) Can you add or remove equations in (b) to change the unique solution you had to a different unique solution?

In each of (i) - (iv) justify your action in words.

(4) (1.1: Geometry) Suppose we want to express the point $(2, 3)$ in $\mathbb{R}^2$ as the solution space of a system of linear equations.

(a) What is the smallest number of equations you would need? Write down such a system.
(b) Can you add one more equation to the system in (a) so that the new system still has the unique solution $(2, 3)$?
(c) What is the maximum number of distinct equations you can add to your system in (a) to still maintain the unique solution $(2, 3)$?
(d) Is there a general form for the equations in (c)?

(5) The following exercises reveal structural properties of the set of solutions to a system of linear equations. The problems are set in $\mathbb{R}^3$, but the results extend to any $\mathbb{R}^n$.

(a) (i) Suppose $\mathbf{p} = (1, 3, 4)$ and $\mathbf{q} = (5, 8, 12)$ are two points in $\mathbb{R}^3$. Show that the line joining $\mathbf{p}$ and $\mathbf{q}$ consists of all points of the form $\lambda\mathbf{q} + (1 - \lambda)\mathbf{p}$ as λ varies over all real numbers. (**Hint:** Think of the line as anchored $\mathbf{p}$ and going in directions $(\mathbf{q} - \mathbf{p})$ and $-(\mathbf{q} - \mathbf{p})$.)

General Statement: The line joining two points $\mathbf{p}$ and $\mathbf{q}$ in $\mathbb{R}^n$ consists of all points of the form $\lambda\mathbf{q} + (1 - \lambda)\mathbf{p}$ as λ varies over all real numbers.

(ii) Suppose $\mathbf{p} = (1, 3, 4)$ and $\mathbf{q} = (5, 8, 12)$ are solutions to the linear system of equations:

$$\begin{aligned}
a_{11}x_1 + a_{12}x_2 + a_{13}x_3 &= \alpha_1 \\
a_{21}x_1 + a_{22}x_2 + a_{23}x_3 &= \alpha_2 \\
a_{31}x_1 + a_{32}x_2 + a_{33}x_3 &= \alpha_3 \\
a_{41}x_1 + a_{42}x_2 + a_{43}x_3 &= \alpha_4
\end{aligned}$$

Check that all points on the line joining $\mathbf{p}$ and $\mathbf{q}$ are also solutions to the above system of equations.

General Statement: If a system of linear equations in n variables has two solutions, then all points on the line joining the two solutions are also solutions to the system. Therefore, if a system of linear equations has at least two solutions, it has infinitely many solutions.

(b) Suppose $\mathbf{p} = (1, 3, 4)$ is a solution to the system of homogeneous equations:

$$\begin{aligned}
a_{11}x_1 + a_{12}x_2 + a_{13}x_3 &= 0 \\
a_{21}x_1 + a_{22}x_2 + a_{23}x_3 &= 0 \\
a_{31}x_1 + a_{32}x_2 + a_{33}x_3 &= 0 \\
a_{41}x_1 + a_{42}x_2 + a_{43}x_3 &= 0
\end{aligned}$$

Check that any multiple of $\mathbf{p}$, i.e., a vector of the form $\lambda(1, 3, 4)$ where λ is any real number, is also a solution of the system. Is this an application of the previous question?

General Statement: If a homogeneous system of equations has a non-zero solution then it has infinitely many solutions.

(c) Consider the linear system of equations $\{x = 1, y = 2\}$ in the three variables x, y, z.

(i) Find the solution set of this system in $\mathbb{R}^3$.

(ii) Now consider the equation $\alpha x + \beta y = \alpha + 2\beta$ obtained by multiplying the first equation by the real number α and the second equation by the real number β and adding the two resulting equations. For instance, if $\alpha = 3$ and $\beta = 2$ you get the equation $3x + 2y = 7$. This is essentially the third elementary operation applied to the two original equations.
Draw the planes in $\mathbb{R}^3$ corresponding to the three equations $x = 1$, $y = 2$, $3x + 2y = 7$. Is the solution set of the new system the same as the solution set of the old system?

(iii) More generally, argue that the solution set of the system $\{x = 1, y = 2, \alpha x + \beta y = \alpha + 2\beta\}$ is the same as the solution set of $\{x = 1, y = 2\}$.

(iv) Now argue that you can remove $x = 1$ from the system $\{x = 1, y = 2, \alpha x + \beta y = \alpha + 2\beta\}$ and get the same solutions as in (i) as long as (α, β) is not a multiple of $(0, 1)$. Similarly, you can remove $y = 2$

from the system $\{x = 1, y = 2, \alpha x + \beta y = \alpha + 2\beta\}$ and get the same solutions as in (i) as long as (α, β) is not a multiple of $(1, 0)$.

(6) (1.1/1.2: Interpolating polynomials) Say we want to find a polynomial $f(x)$ of degree 3,

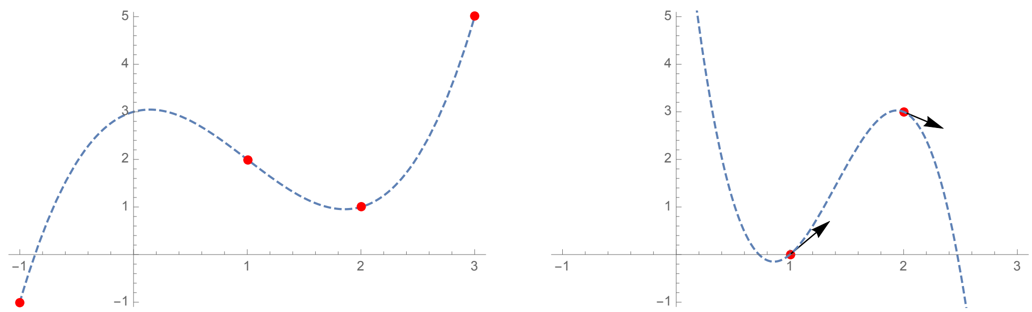
$$f(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3,$$

satisfying some interpolation conditions. In each case below, write a system of linear equations whose solutions are (a_0, a_1, a_2, a_3). You don't need to solve the system.

(a) We want $f(x)$ to pass through the points $(-1, -1), (1, 2), (2, 1)$ and $(3, 5)$.

(b) We want $f(x)$ to pass through $(1, 0)$ with derivative $+2$ and $(2, 3)$ with derivative -1.

Graphically:



(c) (Discuss) What if we had more than four points to consider? Fewer?

(d) (Discuss) Can we still use linear algebra if $f(x)$ is another kind of function, such as $f(x) = a\sin(x) + b\cos(x)$? $f(x) = ae^{bx}$?

(7) (1.2: Solving linear equations) (a) Use Gauss-Jordan elimination to find the general solution for the following system of linear equations:

$$\begin{aligned} z_2 + 3z_3 - z_4 &= 0 \\ -z_1 - z_2 - z_3 + z_4 &= 0 \\ -2z_1 - 4z_2 + 4z_3 - 2z_4 &= 0 \end{aligned}$$

(b) Give an example of a non-zero solution to the previous system of linear equations.

(c) The points $(1, 0, 3)$, $(1, 1, 1)$, and $(-2, -1, 2)$ lie on a unique plane $a_1x_1 + a_2x_2 + a_3x_3 = b$. Using your previous answers, find an equation for this plane. (**Hint:** think about the relationship between the previous system and the one you would need to solve in this question.)

(8) (1.2: Solving linear equations) Consider the linear system

$$\begin{array}{rcrcrcl} 2x_1 & + & 3x_2 & - & 5x_3 & = & b_1 \\ 7x_1 & + & 2x_2 & + & 8x_3 & = & b_2 \\ -x_1 & + & x_2 & - & 5x_3 & = & b_3 \end{array}$$

(a) Find the echelon form of the augmented matrix of the above system.

(b) Find the conditions on b_1, b_2, b_3 for which this system has a solution.

(c) Do you see the shape of the points (b_1, b_2, b_3) for which the above system has a solution?

(d) If you randomly picked a (b_1, b_2, b_3) in $\mathbb{R}^3$, do you expect the above system to have a solution?

(9) (1.2: Solving linear equations) Consider the following linear system with a and b unknown non-zero constants.

$$\begin{array}{rcrcrcr} x_1 & - & 3x_2 & + & x_3 & = & 4 \\ 2x_1 & & & - & 8x_3 & = & -2 \\ -6x_1 & + & 6x_2 & + & ax_3 & = & b \end{array}$$

(a) For what values of a and b does the system have infinitely many solutions?
(b) Given an example of a and b where the system has exactly one solution.
(c) Give an example of a and b for which the system has no solutions.