MATH 308 O
Exam II
May 15, 2020

Name ______________________________

Student ID #______________

HONOR STATEMENT

"I affirm that my work upholds the highest standards of honesty and academic integrity at the University of Washington, and that I have neither given nor received any unauthorized assistance on this exam."

SIGNATURE:______________________________

SOLUTIONS!

1	16	
2	10	
3	12	
4	12	
Bonus	5	
Total	50	

- Your exam should consist of this cover sheet, followed by 4 problems and a bonus question. Check that you have a complete exam.
- Pace yourself. You have 50 minutes to complete the exam and there are 4 problems. Try not to spend more than 10 minutes on each problem.
- Show all your work and justify your answers.
- Your answers should be exact values rather than decimal approximations. (For example, $\frac{\pi}{4}$ is an exact answer and is preferable to its decimal approximation 0.7854.)
- You may use TI-30X IIS calculator and one 8.5×11-inch sheet of handwritten notes. All other electronic devices (including graphing calculators) are forbidden.
- The use of headphones or earbuds during the exam is not permitted.
- There are multiple versions of the exam, you have signed an honor statement, and cheating is a hassle for everyone involved. DO NOT CHEAT.
- Turn your cell phone OFF and put it AWAY for the duration of the exam.

GOOD LUCK!

1. (16 Points) True / False and Short Answer.

Clearly indicate whether the statement is true or false and **justify your answer**.

(a) **TRUE / FALSE** $\mathbb{R}^4$ is a subspace of $\mathbb{R}^6$.

(2) False, $\mathbb{R}^4$ is not a subset of $\mathbb{R}^6$.

OR

$\begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \notin \mathbb{R}^4$.

(b) **TRUE / FALSE** Let A be a 4×3 matrix. Then the nullity$(A) \geq 1$.

(2) False, for example $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$ has rank$(A) = 3$,

So by rank-nullity, nullity$(A) = 0$.

$\left(A = \begin{bmatrix} \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \end{bmatrix}_{4\times 3} \right)$ (alternatively, lin. ind. columns . . .)

(c) **TRUE / FALSE** Let A be a 4×3 matrix. Then $T(\vec{x}) = A\vec{x}$ could be one-to-one.

(2) True, $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$ has linearly-independent columns, or a pivot in each column, hence is one-to-one.

Give an example of each of the following. If there is no such example, write NOT POSSIBLE and **justify why it is not possible.** If you provide an example, you do not need to justify why the example works.

(d) **Give 3 examples** of matrices whose square is the identity, but is not itself the identity. (Find $A \neq I_2$ such that $A^2 = I_2$.)

(3)

$$A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

$$A = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

$$A = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}.$$

(e) **Give an example** of a linear transformation $T : \mathbb{R}^5 \mapsto \mathbb{R}^5$ such that $\text{Range}(T) = \text{Ker}(T)$.

(2)

Not possible

If range(T) = ker(T), then dim(range(T)) = dim(ker(T)). This means rank(A) = nullity(A) for a matrix A representing the transformation. By Rank-Nullity, this means 2rank(A) = 5, not possible for integers.

(f) **Give an example** of a 2×2 matrix that represents first scaling by 2, then rotating by $\frac{\pi}{4}$, then reflecting over the y-axis (in that order).

(3)

$$\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} \cos(\pi/4) & -\sin(\frac{\pi}{4}) \\ \sin(\pi/4) & \cos(\pi/4) \end{bmatrix}\begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} \sqrt{2} & -\sqrt{2} \\ \sqrt{2} & \sqrt{2} \end{bmatrix} = \begin{bmatrix} -\sqrt{2} & \sqrt{2} \\ \sqrt{2} & \sqrt{2} \end{bmatrix}$$

(g) **Give an example** of a 4×3 matrix A such that $T(\vec{x}) = A\vec{x}$ is onto.

(2)

Not possible $T: \mathbb{R}^3 \to \mathbb{R}^4$, and $3 < 4$.

(Alternatively: column space is the span of 3 vectors, so its dimension is at most 3.)

2. (10 points)

(a) Solve the following matrix equation for X. You may assume all matrices are square and invertible. You must show all of your work to receive credit.

$$AX = AXC - BC$$

$$AX - AXC = -BC$$

$$A(X - XC) = -BC$$

$$A^{-1}A(X - XC) = A^{-1}(-BC)$$ (+1)

$$(X - XC) = -A^{-1}BC$$

(2pts) (*) $$(XI - XC) = -A^{-1}BC$$

$$X(I - C) = -A^{-1}BC$$ (+3)

$$X(I-C)(I-C)^{-1} = -A^{-1}BC(I-C)^{-1}$$

$$\boxed{X = -A^{-1}BC(I-C)^{-1}}$$

(b) Solve the following matrix equation for X. You may assume all matrices are square and invertible. You must show all of your work to receive credit.

$$X(AB - CD) = XA + BC$$

$$X(AB-CD) - XA = BC$$

$$X((AB-CD) - A) = BC$$

$$X((AB-CD)-A)((AB-CD)-A)^{-1} = BC((AB-CD)-A)^{-1}$$

$$\boxed{X = BC((AB-CD)-A)^{-1}}$$

3. (12 points)

Let $T : \mathbb{R}^4 \to \mathbb{R}^4$ be the linear transformation given by:

$$T(\vec{x}) = \begin{bmatrix} 2x_1 + x_2 + x_3 + x_4 \\ x_2 + x_3 + x_4 \\ 3x_3 + 3x_4 \\ 2x_4 \end{bmatrix} \tag{1}$$

(a) Show the the set of vectors in $\mathbb{R}^4$ that scale by 2 under the transformation T forms a subspace.

Notice, $T(\vec{x}) = \begin{bmatrix} 2 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 3 & 3 \\ 0 & 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$. Let $A = \begin{bmatrix} 2 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 3 & 3 \\ 0 & 0 & 0 & 2 \end{bmatrix}$

Want to show $S = \{\vec{v} \in \mathbb{R}^4 \mid A\vec{v} = 2\vec{v}\}$ is a subspace.

Then $S = \{\vec{v} \in \mathbb{R}^4 \mid A\vec{v} - 2\vec{v} = \vec{0}\}$

$= \{\vec{v} \in \mathbb{R}^4 \mid A\vec{v} - 2I_4\vec{v} = \vec{0}\}$

$= \{\vec{v} \in \mathbb{R}^4 \mid (A - 2I_4)\vec{v} = \vec{0}\}$

$= \text{null}(A - 2I_4)$.

* There are many alternative solutions!

Since null spaces are subspaces, (thm 5) S is a subspace.

(b) Find a basis for the subspace from part (a). Show all of your work.

$(A - 2I_4)\vec{v} = \vec{0}$, want to find a basis for this null space.

$$\begin{bmatrix} 2 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 3 & 3 \\ 0 & 0 & 0 & 2 \end{bmatrix} - 2\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 1 & 1 \\ 0 & -1 & 1 & 1 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

null space:

$$\left[\begin{array}{cccc|c} 0 & 1 & 1 & 1 & 0 \\ 0 & -1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 3 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] \sim \left[\begin{array}{cccc|c} 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 2 & 2 & 0 \\ 0 & 0 & 1 & 3 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] \sim \left[\begin{array}{cccc|c} 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 3 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] \sim \left[\begin{array}{cccc|c} 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

Let $x_1 = t$, then $x_4 = 0$, $x_3 = 0$, $x_2 = 0$.

$\vec{x} = t\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$, so $\mathcal{B}_{\text{null}(A)} = \left\{\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}\right\}$.

lin. ind. as a set.

4. (12 Points) Let T be a linear transformation such that $T(\vec{x}) = A\vec{x}$ where A and its equivalent reduced row echelon form is given by

$T: \mathbb{R}^4 \to \mathbb{R}^3$

$$A = \underbrace{\begin{bmatrix} 1 & 1 & 4 & 2 \\ 1 & 4 & 1 & -1 \\ 0 & 3 & -3 & -3 \end{bmatrix}}_{3\times 4} \sim \begin{bmatrix} 1 & 0 & 5 & 3 \\ 0 & 1 & -1 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\begin{array}{cc} \uparrow & \uparrow \\ \text{p.c.} & \text{p.c.} \end{array}$$

(a) What is the domain of T? $\mathbb{R}^4$

(b) What is the codomain of T? $\mathbb{R}^3$

(c) Give a basis for the column space of A that includes the vector $\begin{bmatrix} 4 \\ 1 \\ -3 \end{bmatrix}$. If this is not possible, explain why. If it is possible, show all of your work and explain your thinking.

By method 2, $\mathcal{B}_{col(A)} = \left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 4 \\ 3 \end{bmatrix} \right\}$, which means $\dim(col(A)) = 2$, and any basis will contain 2 elements.

* many alternative solutions!

Since $col(A) = span\left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 4 \\ 3 \end{bmatrix}, \begin{bmatrix} 4 \\ 1 \\ -3 \end{bmatrix}, \begin{bmatrix} 2 \\ -1 \\ -3 \end{bmatrix} \right\}$, we can see that $\begin{bmatrix} 4 \\ 1 \\ -3 \end{bmatrix} \in col(A)$ since $0 \cdot \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + 0 \cdot \begin{bmatrix} 1 \\ 4 \\ 3 \end{bmatrix} + 1 \cdot \begin{bmatrix} 4 \\ 1 \\ -3 \end{bmatrix} + 0 \begin{bmatrix} 2 \\ -1 \\ -3 \end{bmatrix} = \begin{bmatrix} 4 \\ 1 \\ -3 \end{bmatrix}$.

Now, notice that in the matrix $\begin{bmatrix} 1 & 0 & 5 & 3 \\ 0 & 1 & -1 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$, the first and third columns are linearly independent: $\left[\begin{array}{cc|c} 1 & 5 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{array}\right]$.

$\uparrow\uparrow$ all columns have pivots

Thus, by a theorem, the first and third columns in A are linearly independent since these are equivalent matrices. Then $\left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 4 \\ 1 \\ -3 \end{bmatrix} \right\}$ is lin. ind, and contains 2 vectors, so by a theorem, $\mathcal{B}_{col(A)} = \left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 4 \\ 1 \\ -3 \end{bmatrix} \right\}$, is a basis for $col(A)$.

(d) Is T onto? Justify your answer.

No. $rank(A) = \dim(col(A)) \neq 3$, but $\dim(\mathbb{R}^3) = 3$.

Bonus: (5 points) Find all possible matrices B such at $AB = I_2$ where

$$A = \begin{bmatrix} 1 & 0 & 1 \\ 3 & 3 & 1 \end{bmatrix}$$

(All such matrices would be called "right inverses" of the matrix A. Working this problem shows that they are not unique when A is not square!)

Let $B = \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \\ b_{31} & b_{32} \end{bmatrix}$. Then AB is..

$$\begin{bmatrix} 1 & 0 & 1 \\ 3 & 3 & 1 \end{bmatrix} \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \\ b_{31} & b_{32} \end{bmatrix} = \begin{bmatrix} b_{11} + b_{31} & b_{12} + b_{32} \\ 3b_{11} + 3b_{21} + b_{31} & 3b_{12} + 3b_{22} + b_{32} \end{bmatrix} \overset{\text{(And we want)}}{=} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

So: $\begin{cases} b_{11} + b_{31} = 1 \\ 3b_{11} + 3b_{21} + b_{31} = 0 \end{cases}$ and $\begin{cases} b_{12} + b_{32} = 0 \\ 3b_{12} + 3b_{22} + b_{32} = 1 \end{cases}$

could justify starting here!

$$\begin{array}{ccc|c} b_{11} & b_{21} & b_{31} & \\ \end{array}$$
$$\left[\begin{array}{ccc|c} 1 & 0 & 1 & 1 \\ 3 & 3 & 1 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 0 & 1 & 1 \\ 0 & 3 & -2 & -3 \end{array}\right]$$

f.v.

$$\begin{array}{ccc|c} b_{12} & b_{22} & b_{32} & \\ \end{array}$$
$$\left[\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 3 & 3 & 1 & 1 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 3 & -2 & 1 \end{array}\right]$$

f.v.

$b_{31} = s$

$b_{21} = -1 + \frac{2}{3}s$

$b_{11} = 1 - s$

$b_{32} = t$

$b_{22} = \frac{1}{3} + \frac{2}{3}t$

$b_{12} = -t$

All of the form:

$$B = \begin{bmatrix} 1-s & -t \\ -1 + \frac{2}{3}s & \frac{1}{3} + \frac{2}{3}t \\ s & t \end{bmatrix}$$ for any s,t real numbers.