MATH 308 O
Exam I
April 22, 2020

Name ____

Student ID #____________________

HONOR STATEMENT

"I affirm that my work upholds the highest standards of honesty and academic integrity at the University of Washington, and that I have neither given nor received any unauthorized assistance on this exam."

SIGNATURE:____________________________________

SOLUTIONS !

1	20	
2	10	
3	10	
4	10	
Bonus	5	
Total	50	

- Your exam should consist of this cover sheet, followed by 4 problems and a bonus question. Check that you have a complete exam.
- Pace yourself. You have 50 minutes to complete the exam and there are 4 problems. Try not to spend more than 10 minutes on each problem. You will have 10 minutes at the end of the exam to upload your solutions to Gradescope.
- Show all your work and justify your answers.
- Your answers should be exact values rather than decimal approximations. (For example, $\frac{\pi}{4}$ is an exact answer and is preferable to its decimal approximation 0.7854.)
- This is an open book exam, however, you are not allowed to collaborate with anyone.
- There are multiple versions of the exam, you have signed an honor statement, and cheating is a hassle for everyone involved. DO NOT CHEAT.
- Turn your cell phone OFF and put it AWAY for the duration of the exam.

GOOD LUCK!

1. (18 Points) True / False and Short Answer (2 pages).

   Clearly indicate whether the statement is true or false. **If true, justify your answer. If false, provide a counterexample or give justification.**

   (a) **TRUE / FALSE** A set of vectors $\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ is linearly ~~in~~dependent if and only if the first vector in the set, $\vec{v}_1$, is in the span of the others vectors, $\{\vec{v}_2, \vec{v}_3\}$.

   False Counterexample $\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}$

   (b) **TRUE / FALSE** The zero vector, $\vec{0}$, can be a solution to a nonhomogeneous linear system of equations.

   False $A\vec{0} = \vec{0}$ for any matrix A.

   (c) **TRUE / FALSE** A system of equations with more variables than equations always has at least one solution.

   False $\begin{cases} x_1 + x_2 + x_3 = 0 \\ x_1 + x_2 + x_3 = 1 \end{cases}$

Give an example of each of the following (and answer the question if needed). If there is no such example, write NOT POSSIBLE. You **do not** need to justify that your example satisfies the desired conditions, nor do you need to justify answers to the additional questions.

(d) **Give an example** of a linear system of equations with exactly four variables with no solutions such that when you remove an equation (you can pick which one), the remaining system of equations has exactly one solution.

$$\begin{cases} x_1 + x_2 + x_3 + x_4 = 1 & \leftarrow \text{remove.} \\ x_1 + x_2 + x_3 + x_4 = 0 \\ x_2 + x_3 + x_4 = 0 \\ x_3 + x_4 = 0 \\ x_4 = 0 \end{cases}$$

(e) **Give an example** of a set of vectors that is linearly independent but does not span $\mathbb{R}^2$. How many vectors can be in this set? Are there any other conditions?

$\left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right\}$. The set must have exactly one vector and it must not be the zero vector.

(f) **Give an example** of a set of vectors that spans $\mathbb{R}^4$ but is not linearly independent. What is the minimum number of vectors that must be in the set?

$$\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} \right\}$$

The set must include at least 5 vectors.

2. (10 Points) Identify which matrices are equivalent under row operations. Justify by listing the row operations (using notation from the lecture notes or clearly expressing each operation in words). Then, for matrices that are not equivalent, explain how you know.

$$\left\{ \begin{bmatrix} 3 & -6 & 9 \\ 0 & 2 & 8 \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 3 & 3 & 2 \\ 6 & 6 & 4 \\ 1 & 0 & 4 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 4 \\ 0 & 3 & -10 \\ 0 & 0 & 0 \end{bmatrix} \right\}$$

$$\begin{bmatrix} 3 & -6 & 9 \\ 0 & 2 & 8 \\ 0 & 0 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & -2 & 3 \\ 0 & 1 & 4 \\ 0 & 0 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & -2 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$\frac{1}{3}R_1 \to R_1$, $\frac{1}{2}R_2 \to R_2$ $\qquad$ $-4R_3 + R_2 \to R_2$, $-3R_3 + R_1 \to R_1$ $\qquad$ $2R_2 + R_1 \to R_1$

$$\begin{bmatrix} 3 & 3 & 2 \\ 6 & 6 & 4 \\ 1 & 0 & 4 \end{bmatrix} \sim \begin{bmatrix} 3 & 3 & 2 \\ 0 & 0 & 0 \\ 1 & 0 & 4 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 4 \\ 0 & 0 & 0 \\ 3 & 3 & 2 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 4 \\ 0 & 0 & 0 \\ 0 & 3 & -10 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 4 \\ 0 & 3 & -10 \\ 0 & 0 & 0 \end{bmatrix}$$

$-2R_1 + R_2 \to R_2$ $\qquad$ $R_1 \leftrightarrow R_3$ $\qquad$ $-3R_1 + R_3 \to R_3$ $\qquad$ $R_3 \leftrightarrow R_2$

So, $\begin{bmatrix} 3 & -6 & 9 \\ 0 & 2 & 8 \\ 0 & 0 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ and $\begin{bmatrix} 3 & 3 & 2 \\ 6 & 6 & 4 \\ 1 & 0 & 4 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 4 \\ 0 & 3 & -10 \\ 0 & 0 & 0 \end{bmatrix}$

under the row operations above. Notice that $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \not\sim \begin{bmatrix} 1 & 0 & 4 \\ 0 & 3 & -10 \\ 0 & 0 & 0 \end{bmatrix}$ since one matrix has 3 pivots and the other 2 pivots, and row operations preserve solution sets to $A\vec{x} = \vec{0}$.

(Alternatively, we could say that $\begin{bmatrix} 1 & 0 & 4 \\ 0 & 3 & -10 \\ 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 4 \\ 0 & 1 & -\frac{10}{3} \\ 0 & 0 & 0 \end{bmatrix}$, which is in reduced row echelon form. Since $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ is also in reduced row echelon form and each matrix is equivalent to a unique reduced row echelon form matrix, we see that $\begin{bmatrix} 1 & 0 & 4 \\ 0 & 3 & -10 \\ 0 & 0 & 1 \end{bmatrix} \not\sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$.

3. (12 points) Recall that quadratic polynomials are polynomials of the form $y = a_0 + a_1 x + a_2 x^2$, where a_0, a_1 and a_2 are any real numbers.

(a) For what values of c are there infinitely many different quadratic polynomials such that $(2,0)$, $(3,0)$, $(c,0)$ are points on the quadratic polynomial?

plug points into $y = a_0 + a_1 x + a_2 x^2$:

$$\begin{cases} 0 = a_0 + 2a_1 + 4a_2 & \leftarrow (2,0) \\ 0 = a_0 + 3a_1 + 9a_2 & \leftarrow (3,0) \\ 0 = a_0 + ca_1 + c^2 a_2 & \leftarrow (c,0) \end{cases}$$

Solve for possible values of a_0, a_1, a_2 to get a quadratic polynomial:

$$\left[\begin{array}{ccc|c} 1 & 2 & 4 & 0 \\ 1 & 3 & 9 & 0 \\ 1 & c & c^2 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 2 & 4 & 0 \\ 0 & 1 & 5 & 0 \\ 0 & c-2 & c^2-4 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} & & & \\ & & & \\ 0 & 0 & 0 & 0 \end{array}\right]$$

want infinitely many solutions (∞ many different polynomials)

so we want a zero row. (no pivot in the last row)

To do this, we need $K(R_2) + R_3 \rightarrow R_3 = 0\ 0\ 0$ for some scalar K.

So: $\begin{cases} K(1) = c-2 \\ K\cdot(5) = c^2-4 \end{cases}$ plug k into 2nd equation

$$(c-2)(5) = c^2 - 4$$
$$0 = c^2 - 5c + 6$$
$$0 = (c-2)(c-3), \quad \boxed{c = 2 \text{ or } 3}$$

(b) Using the same points as in part (a), for what values of c does there exist exactly one polynomial passing through the points? It turns out to be the same polynomial for all of these values. What is that polynomial?

For any $c \neq 2$ or 3, there is either 0 or 1 solution.

Since the system is homogeneous, there will be $\propto$ one, and we know which one: trivial solution $\begin{cases} a_0 = 0 \\ a_1 = 0 \\ a_2 = 0 \end{cases}$

Thus, $y = 0 + 0 \cdot x + 0 \cdot x^2 = 0$, so the polynomial is $y = 0$.

4. (10 points) The following system of equations has one unique solution.

$$\begin{cases} x_1 + x_3 = 1 \\ x_2 - x_4 = -7 \\ x_1 - x_2 + x_4 = 15 \\ x_2 + x_3 + x_4 = -3 \end{cases}$$

(a) Identify two methods you can use to show the system has one unique solution, neither of which involves actually solving the system. (Neither of your methods can be "solving the system using an augmented matrix.") Justify that your methods will work.

(Many solutions!)

Method 1: Convert the system into the form $A\vec{x} = \begin{bmatrix} 1 \\ -7 \\ 15 \\ -3 \end{bmatrix}$. Show that the columns of A are linearly independent. This will show there is one unique solution since we can apply the Unifying Theorem (we have 4 vectors in $\mathbb{R}^4$!) (Notice also: 4 equations and 4 variables.)

Method 2: Convert the system into the form $A\vec{x} = \begin{bmatrix} 1 \\ -7 \\ 15 \\ -3 \end{bmatrix}$ and show that the columns of A span $\mathbb{R}^4$. Then, by the unifying theorem, we will have one unique solution. (same justification as above for using Unifying Theorem!)

(b) Use one of these two methods to show the system has one unique solution. (No points will be awarded for directly solving the system.)

Using method 1:

$$\left[\begin{array}{cccc|c} 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & -1 & 0 \\ 1 & -1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 & 0 \end{array}\right] \sim \left[\begin{array}{cccc|c} 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & -1 & 0 \\ 0 & -1 & -1 & 1 & 0 \\ 0 & 1 & 1 & 1 & 0 \end{array}\right] \sim \left[\begin{array}{cccc|c} 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & -1 & 0 \\ 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 2 & 0 \end{array}\right] \sim \left[\begin{array}{cccc|c} 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & -1 & 0 \\ 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 2 & 0 \end{array}\right]$$

Only have the trivial solution (triangular system!), so we see the column vectors are linearly independent. By the unifying theorem, we see that $A\vec{x} = \vec{b}$ will have one unique solution for every $\vec{b}$, and we are done!

BONUS: (5 points) Let A be a matrix whose columns are not linearly independent. If we solve for all solutions to $A\vec{x} = 0$, the solution will be of the form:

$$\vec{x} = s_1\vec{v}_1 + s_2\vec{v}_2 + \cdots + s_n\vec{v}_n$$

for some number of parameters $s_1, s_2, \cdots, s_n$. Show that (with justification) the vectors $\vec{v}_1, \vec{v}_2, \cdots, \vec{v}_n$ are linearly independent.

Hint: Look at the solution to a concrete example, like the homogeneous system in Example 3 in lecture notes 2.3. Are the vectors in the solution linearly independent? Why? Can you generalize that argument to answer this question?

(Many variations)

If $\vec{x} = s_1\vec{v}_1 + s_2\vec{v}_2 + \cdots + s_n\vec{v}_n$, then $\vec{x} = \begin{bmatrix} x_1 \\ \vdots \\ x_m \end{bmatrix}$

and exactly n of the x_m's are free variables, i.e. ($x_{j_1} = s_1$ for some j_1, $x_{j_2} = s_2$ for some j_2, $j_1 \neq j_2$, and so forth, up to x_{j_n}).

Then $\vec{v}_1$ looks like:

$s_1 \begin{bmatrix} \\ 1 \\ \\ 0 \\ \\ \end{bmatrix}$ ← at x_{j_1}, ← for any other x_{j_k}, $k \neq 1$.

(and numbers possibly in the othe entries..)

More generally, $\vec{v}_k$ for $1 \leq k \leq n$, looks like

$s_k \cdot \begin{bmatrix} 1 \\ \\ \\ 0 \end{bmatrix}$ ← at x_{j_k}, ← for any x_{j_ℓ} where $\ell \neq k$.

"no $\vec{v}_k$ in the set"

Then, any $\vec{v}_k \notin \text{span}\{\vec{v}_1, \vec{v}_2, \dots, \hat{\vec{v}_k}, \dots, \vec{v}_n\}$ since the x_{j_k}-entry in each of the vectors is 0. (and it is 1 in $\vec{v}_k$).

This means we cannot multiply ~~any~~ any of the vectors in the set to give us a 1 in this entry! By a theorem, this means the set is lin. ind. !