Quiz 3 Answer Key

1. Let $y_1, y_2 \in Y$ such that $y_1 \neq y_2$ and let $f: X \to Y$ be a map.

- Contrapositive proof - (see below for a direct proof)

Assume that $f^{-1}(\{y_1\}) \cap f^{-1}(\{y_2\}) \neq \emptyset$. Then, there exists some $x \in f^{-1}(\{y_1\}) \cap f^{-1}(\{y_2\})$. By definition of the intersection, $x \in f^{-1}(\{y_1\})$ and $x \in f^{-1}(\{y_2\})$. Using the definition of the preimage, we see that $f(x) \in \{y_1\}$ and $f(x) \in \{y_2\}$. Since each set $\{y_1\}$ and $\{y_2\}$ contains only one element, we see that $f(x) = y_1$ and $f(x) = y_2$. Then, $y_1 = y_2$ since f is a map. (Remember, maps assign one element to each element in the domain, not two!) $\blacksquare$

- Direct proof -

By the definition of the preimage, we have

$$f^{-1}(\{y_1\}) = \{x \in X : f(x) \in \{y_1\}\} \text{ and}$$

$$f^{-1}(\{y_2\}) = \{x \in X : f(x) \in \{y_2\}\}.$$

Since each set $\{y_1\}$ and $\{y_2\}$ has only one element,

$$f^{-1}(\{y_1\}) = \{x \in X : f(x) = y_1\} \text{ and}$$

$$f^{-1}(\{y_2\}) = \{x \in X : f(x) = y_2\}.$$

Taking an intersection (using the definition of the intersection), we see

$$f^{-1}(\{y_1\}) \cap f^{-1}(\{y_2\}) = \{x \in X : f(x) = y_1 \text{ and } f(x) = y_2\}.$$

Since f is a map, and $y_1 \neq y_2$, there are no such $x \in X$. Thus, the set is empty, as desired. ▨

2. We will prove directly that the set $\{1,2,3\}$ has exactly 5 equivalence relations. First, observe that the set $\{1,2,3\}$ has size 3 (using the identity map on the set). The identity map is a bijection by Theorem 1. Then, by theorem 2, $\{1,2,3\}$ has 5 partitions. By THM 5 from week 8, we know that each partition gives rise to an equivalence relation. Thus, the set $\{1,2,3\}$ has at least 5 equivalence relations. By THM 6 from week 8, we know that each equivalence relation gives rise to a partition where the rule for assigning parts is identical to the rule in THM 5. Thus, we have at most 5 equivalence relations. Hence, there are exactly 5 equivalence relations. ▨

3. (Bonus)

Assume $X \sim \mathbb{N}$ and $\mathbb{N} \sim Y$. By definition of cardinal equivalence, there is a bijection $f: X \to \mathbb{N}$ and a bijection $g: \mathbb{N} \to Y$. We can compose these maps using the definition of a composite map to get a map $g \circ f: X \to Y$. By <u>THM 4</u> from week 5, $g \circ f$ is a bijection since g and f are each bijections.

Since $g \circ f$ is a bijection, an inverse map exists, so $(g \circ f)^{-1}: Y \to X$. By theorem 3, $(g \circ f)^{-1}$ is a bijection. By definition of cardinal equivalence, we can conclude $Y \sim X$. ☒

4. (Bonus)

Let $X = \{1, 2, \dots, n\}$ and $Y = \{1, 2, \dots, m\}$ and assume $m > n$. By the pigeonhole principle, Lemma 1, there does not exist an injective map $g: Y \to X$. By <u>THM 3</u> from week 8, there exists a surjective map $f: X \to Y$ if and only if there exists an injective map $g: Y \to X$. The contrapositive of both directions of the implication in THM 3 tells us that there does not exist an injective map $g: Y \to X$ if and only if there does not exist a surjective map $f: X \to Y$. Since there does not exist an injective map $g: Y \to X$ (shown above), we can conclude that there does not exist a surjective map $f: X \to Y$, as desired.

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