Written Quiz 2 Answer Key

1. Assume A and B are subsets of a set X.

(a) $A \setminus B = \{x : x \in A \text{ and } x \notin B\}$, by definition 1,

$= \{x : x \in A \text{ and } x \in B^c\}$, by definition of the complement,

$= \{x : x \in A \cap B^c\}$, by defintion of intersection,

$= A \cap B^c$, thus $A \setminus B = A \cap B^c$.

(b) $A \Delta B = A \setminus B \cup B \setminus A$, by definition 2,

$= B \setminus A \cup A \setminus B$, by commutativity of the union,

$= B \Delta A$, by definition 2. Thus, $A \Delta B = B \Delta A$.

(c) Assume $f : Y \to X$ is a map, and A, B are subsets of a set X.

Then,

① $f^{-1}(A \Delta B) = f^{-1}(A \setminus B \cup B \setminus A)$, by definition 2

$= f^{-1}(A \setminus B) \cup f^{-1}(B \setminus A)$, by a theorem,

$= f^{-1}(A \cap B^c) \cup f^{-1}(B \cap A^c)$, by part (a),

$= (f^{-1}(A) \cap f^{-1}(B^c)) \cup (f^{-1}(B) \cap f^{-1}(A^c))$, by a theorem,

$= (f^{-1}(A) \cap f^{-1}(B)^c) \cup (f^{-1}(B) \cap f^{-1}(A)^c)$, by a theorem.

Now, notice

$f(f^{-1}(A \Delta B)) = f((f^{-1}(A) \cap f^{-1}(B)^c) \cup (f^{-1}(B) \cap f^{-1}(A)^c))$, by ①,

$= f(f^{-1}(A) \cap f^{-1}(B)^c) \cup f(f^{-1}(B) \cap f^{-1}(A)^c)$, by a theorem,

$\subset [f(f^{-1}(A)) \cap f(f^{-1}(B)^c)] \cup [f(f^{-1}(B)) \cap f(f^{-1}(A)^c)]$,

by a theorem.

$\boxtimes$

2. We use induction.

<u>Base Case</u>: Let $n=1$. $1^3=1$ and $\left(\frac{1(1+1)}{2}\right)^2=1$, so the statement holds for $n=1$.

<u>Inductive Step</u>: Assume the sum of the cubes of the first n integers is $\left(\frac{n(n+1)}{2}\right)^2$. Consider

sum of the cubes of the first $(n+1)$ integers.

$$\sum_{i=1}^{n+1} i^3 = \left(\sum_{i=1}^{n} i^3\right) + (n+1)^3$$

$$= \left(\frac{n(n+1)}{2}\right)^2 + (n+1)^3, \text{ by the inductive hypothesis,}$$

$$= \left(\frac{(n+1)(n+2)}{2}\right)^2, \text{ by theorem 1,}$$

which completes the inductive step.

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