Written Quiz 1 Key

1. (Example direct method. An example contradiction proof follows.)

Assume m, n and N are odd integers. Notice that N is the sum of two integers since integers are closed under multiplication. (In turn, we know N is an integer since integers are closed under addition, but we already knew that N is an integer.) Since N is odd, Theorems 2 and 3 tell us that mk and $n\ell$ cannot both be odd nor both even. Combining this with Theorem 4 (or using the definition of odd to conclude all integers are either even or odd, not both), we see that exactly one of mk and $n\ell$ is even (and the other is odd).

We need the following facts.

1. The product of two odd numbers is odd.
2. The product of an odd and even number is even.

To see 1, assume $\tilde{m}$ and $\tilde{n}$ are odd integers. By

Theorem 1, we know $\tilde{m} = 2\tilde{k}+1$ for some integer $\tilde{k}$ and $\hat{n} = 2\hat{\ell}+1$ for some integer $\hat{\ell}$. Then $\tilde{m}\tilde{n} = 4\tilde{k}\hat{\ell} + 2\tilde{k} + 2\hat{\ell} + 1$ $= 2(2\hat{k}\hat{\ell} + \hat{k} + \hat{\ell}) + 1$. Since $2\tilde{k}\hat{\ell} + \hat{k} + \hat{\ell}$ is an integer, by Theorem 1, we know $\tilde{m}\tilde{n}$ is odd.

To see 2, let $\tilde{m}$ be odd and $\tilde{n}$ be even. Then $\tilde{n} = 2\hat{k}$ for some integer $\hat{k}$, and $\tilde{m}\tilde{n} = \tilde{m}(2\hat{k}) = 2\tilde{m}\hat{k}$. Since 2 is a divisor of $\tilde{m}\tilde{n}$, $\tilde{m}\tilde{n}$ is even by defintion.

Now, we can finish the proof. Since we have assumed m and n are odd, and we have already shown that exactly one of mk and $n\ell$ is even, we can use facts 1 and 2 above to conclude that exactly one of k and ℓ is even, as desired. $\boxtimes$

(Example Contradiction)

1. Assume either k and ℓ are both odd or that k and ℓ are both even, and that $m, n,$ and N are odd.

Case 1: Assume k and ℓ are both odd. Since we are assuming $m, n, k,$ are odd, we can use Theorem 1 to write

$$m = 2n_1 + 1,$$
$$n = 2n_2 + 1,$$
$$k = 2n_3 + 1,$$
$$\ell = 2n_4 + 1,$$

for integers n_1, n_2, n_3, n_4. Computing, we see

$$\begin{aligned} N &= mk + n\ell \\ &= (2n_1+1)(2n_3+1) + (2n_2+1)(2n_4+1) \\ &= 4n_1n_3 + 2n_1 + 2n_3 + 1 + 4n_2n_4 + 2n_2 + 2n_4 + 1 \\ &= 2(2n_1n_3 + 2n_2n_4 + n_1 + n_2 + n_3 + n_4 + 1). \end{aligned}$$

Since $2n_1n_3 + 2n_2n_4 + n_1 + n_2 + n_3 + n_4 + 1$ is an integer, we see that N has 2 as a divisor, hence N is even by definition. This is a contradiction since we assumed N to be odd.

Case 2: Assume k and ℓ are both even. Since we are also assuming m and n are odd, we can use the definition of even and Theorem 1 to write

$$m = 2n_1 + 1,$$
$$n = 2n_2 + 1,$$
$$k = 2n_3,$$
$$\ell = 2n_4,$$

for some integers n_1, n_2, n_3, and n_4. Computing, we see

$$\begin{aligned} N &= (2n_1+1)\,2n_3 + (2n_2+1)\,2n_4 \\ &= 4n_1n_3 + 2n_3 + 4n_2n_4 + 2n_4 \\ &= 2(2n_1n_3 + 2n_2n_4 + n_3 + n_4), \end{aligned}$$

and as before, since $2n_1n_3 + 2n_2n_4 + n_3 + n_4$ is an integer, we see that 2 is a divisor of N and that N is even by definition. This is a contradiction since we assumed N to be odd.

$\blacksquare$

2. Assume otherwise, in other words, that $\frac{1}{e}$ is rational. By definition, $\frac{1}{e} = \frac{p}{q}$ for some integers p and q such that $q \neq 0$. Additionally, notice that $p \neq 0$, otherwise $\frac{1}{e} = 0$, which implies $1 = 0$ (which is false!). Since $p \neq 0$, we may multiply each side of the equation above by $\frac{q}{p}$ and by e, giving us $e = \frac{q}{p}$, where q and p are integers and $p \neq 0$. In other words, e is rational by definition. However, this contradicts Theorem 5.

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