Lecture Notes: Week #8

$\mathbb{R}$ is uncountable; Partitions + Equivalence Classes

We will continue in these notes where we left off in the last set of notes.

THM 4 $\mathbb{R}$ is uncountable.

To show this, we will need a Lemma.

LEMMA Each real number x such that $0 \le x < 1$ has a unique decimal expansion of the form

$$x = 0.d_1 d_2 d_3 \ldots$$

where each $d_i \in \{0, 1, 2, \ldots, 9\}$, provided that we disallow decimal expansions ending in an infinite string of 9's.

(e.g. $0.349999\ldots = 0.350000\ldots$)

NOTE We are allowing each decimal expansion to converge to exactly one $x \in [0, 1)$.

NOTE We have not talked about what it means to converge. You will see this in Math 327, and various generalizations in analysis and topology courses. However, in this proof, the idea of convergence its absolutely necessary. As such, you are not responsible for fully understanding the following sketch of a proof, but you should strive to understand at least the idea of the proof.

Sketch: (of proof of THM $\tilde{4}$)

The proof here comes in several pieces. We must first show that decimal expansions make sense, that is, $0.d_1d_2d_3\ldots$ really does converge to a real number, and it is really in $[0,1)$. Here, we will need to remember that a decimal expansion can be represented as follows:

$$0.d_1d_2d_3\ldots = \frac{d_1}{10} + \frac{d_2}{10^2} + \frac{d_3}{10^3} + \cdots.$$

Second, we need to show that given any $x \in [0,1)$, there really is a decimal expansion (potentially infinitely long) that represents x. Then, lastly, we must show that this expansion is unique (which we will be able to do since the Lemma disallows infinite strings of nines).

① We show that $0.d_1 d_2 d_3 \ldots$ makes sense.

A decimal expansion, by definition, is the following:

$$0.d_1 d_2 d_3 \ldots = \frac{d_1}{10} + \frac{d_2}{10^2} + \frac{d_3}{10^3} + \cdots$$

We must show that the right side will converge regardless of our choice of d_i.

The idea is the following: allow each d_i to be as large as possible, show that this will converge, which will imply that anything else we choose (necessarily smaller) will also converge. (This is called the Comparison test.) Here, we will pick $d_i = 9$ for all i. This is an infinite string of nines, so we are disallowing it, however, if it converges, everything that we are allowing will also converge.

Notice

$$\frac{9}{10} + \frac{9}{10^2} + \frac{9}{10^3} + \cdots = \frac{9}{10}\left(1 + \frac{1}{10} + \frac{1}{10^2} + \cdots\right)$$

geometric sum!

$$= \frac{9}{10}\left(\sum_{k=0}^{\infty}\left(\frac{1}{10}\right)^k\right)$$

which we have a formula for computing!

$$= \frac{9}{10}\left(\frac{1}{1-\frac{1}{10}}\right)$$

$$= \frac{9}{10}\left(\frac{1}{\frac{9}{10}}\right)$$

$$= 1.$$

Thus, our decimal expansions converge. Further, we see that since we are disallowing infinite strings of nines, all of our decimal expansions will be strictly less than 1, i.e. they are real numbers in $[0,1)$. Do our decimal expansions miss any real number? i.e. If $x \in [0,1)$, does it have a decimal expansion?

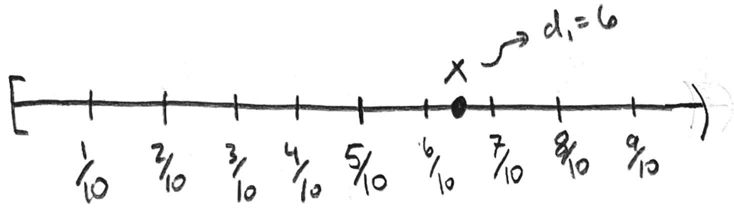
(2) Next, we will show that any x in $[0,1)$ has

a (convergent) decimal expansion. Here's the idea:

Let $x \in [0,1)$. Then $x \in \left[\frac{k}{10}, \frac{k+1}{10}\right)$ for some $k \in \{0,1,\dots,9\}$.

Take $d_1 = k$.

PIC



$x = \frac{6}{10} + r_1$ ← some remainder

Then $x = \dfrac{d_1}{10} + r_1$, where r_1 is a remainder such that $0 \le r_1 < \frac{1}{10}$. This means that $0 \le 10r_1 < 1$, so $10r_1$ can be expanded the same way we just expanded x. Namely, there is a $d_2 \in \{0,1,\dots,9\}$ such that

$$10r_1 = \frac{d_2}{10} + r_2, \quad \text{for } 0 \le r_2 < \tfrac{1}{10}$$

hence,

$$r_1 = \frac{d_2}{10^2} + s_2, \quad \text{for } 0 \le s_2 < \frac{1}{10^2}.$$

Now we can expand s_2 similarly: $0 \le 10^2 s_2 < 1$.

Continuing in this way (i.e. rigorously, use induction)

we see

$$X = \frac{d_1}{10} + \frac{d_2}{10^2} + \frac{d_3}{10^3} + \cdots + \frac{d_n}{10^n} + s_n,$$

where each $d_i \in \{0, 1, \ldots, 9\}$ for $i \in \{1, \ldots, n\}$, and

$$0 \le s_n < \frac{1}{10^n}.$$

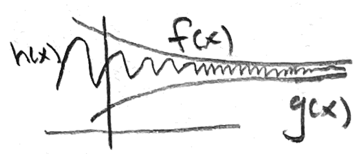
Now, recall the squeeze theorem from Math 124:

if $\lim_{x\to\infty} f(x) = a$ and $\lim_{x\to\infty} g(x) = a$, and we

know $g(x) \le h(x) \le f(x)$ for all x, then

$\lim_{x\to\infty} h(x) = a$.

<u>PIC</u>



Something very similar is happening here with s_n (except we have a discrete set of values). Since s_n is stuck between 0 and $\frac{1}{10^n}$, and $\lim_{n\to\infty} 0 = 0$, and

$\lim_{n\to\infty} \frac{1}{10^n} = 0$, we see that $\lim_{n\to\infty} s_n = 0$.

Now, if we think of s_n as our "error term", i.e. how far from x our finte decimal expansion is, we see that if we continue the expansion forever (potentially making it infinitely long) the decimal expansion converges to x, meaning we have a decimal expansion for x. Is it unique?

(3) We now show that the expansion is unique. Sketch! Say we have two different expansions for the same x, call them $x = d_1$ and $x = d_2$. Then $x - x = 0$, so $d_1 - d_2 = 0$. Since we are not allowing infinite strings of nines, this means that d_1 and d_2 must have at least one digit different, meaning there difference is positive, not 0.

$\boxtimes$

Now, we can use this LEMMA to prove THM $\tilde{4}$.

THM $\tilde{4}$ $\mathbb{R}$ is uncountable.

proof: Assume otherwise, that is, that $\mathbb{R}$ is countable for the sake of contradiction. Since $[0,1) \subset \mathbb{R}$ is an infinite subset, by THM 3 ③ (from last week), $[0,1)$ is also countable.

Let $f: [0,1) \longrightarrow \mathbb{N}$ be a bijection. Then $f^{-1}: \mathbb{N} \longrightarrow [0,1)$ is also a bijection (from an earlier homework problem). Now, label:

$$f^{-1}(1) = x_1$$ (we are labeling the value of $f^{-1}(1)$ by x_1.)

$$f^{-1}(2) = x_2$$
$$f^{-1}(3) = x_3$$
$$\vdots$$
$$f^{-1}(k) = x_k$$
$$\vdots$$

Now, by the LEMMA, each x_k has a unique decimal expansion. We can list them / label them as follows:

$$x_1 = 0.d_{11}\, d_{12}\, d_{13}\, d_{14}\, d_{15} \cdots$$
$$x_2 = 0.d_{21}\, d_{22}\, d_{23}\, d_{24}\, d_{25} \cdots$$
$$x_3 = 0.d_{31}\, d_{32}\, d_{33}\, d_{34}\, d_{35} \cdots$$
$$x_4 = 0.d_{41}\, d_{42}\, d_{43}\, d_{44}\, d_{45} \cdots$$
$$x_5 = 0.d_{51}\, d_{52}\, d_{53}\, d_{54}\, d_{55} \cdots$$
$$\vdots \qquad\qquad \ddots$$

Now, define a number $x^* \in [0,1)$ using the following decimal expansion.

$$x^* = 0.d_1^*\, d_2^*\, d_3^*\, d_4^*\, d_5^* \cdots$$

we avoid nine to keep from having an infinite string appear.

where $d_i^* \in \{0,1,2,\ldots,8\}$ such that

$$\boxed{d_i^* \neq d_{ii}}\,.$$

(we pick an infinte decimal expansion such that it differs from the diagonal!)

$\downarrow$

$$
\begin{aligned}
x_1 &= 0.d_{11}\ d_{12}\ d_{13}\ d_{14}\ d_{15} \cdots \\
x_2 &= 0.d_{21}\ d_{22}\ d_{23}\ d_{24}\ d_{25} \cdots \\
x_3 &= 0.d_{31}\ d_{32}\ d_{33}\ d_{34}\ d_{35} \cdots \\
x_4 &= 0.d_{41}\ d_{42}\ d_{43}\ d_{44}\ d_{45} \cdots \\
x_5 &= 0.d_{51}\ d_{52}\ d_{53}\ d_{54}\ d_{55}
\end{aligned}
$$

Now, observe. $x^* \neq x_1$ since $d_1^* \neq d_{11}$. Similarly, $x^* \neq x_2$ since $d_2^* \neq d_{22}$. In fact, for any $n \in \mathbb{N}$, $x^* \neq x_n$ since $d_n^* \neq d_{nn}$. This means that x^* is <u>not</u> in our list, i.e there is no $n \in \mathbb{N}$ such that $f^{-1}(n) = x^*$, so f^{-1} is not surjective, a contradiction. Thus, $\mathbb{R}$ is uncountable.

$\blacksquare$

<u>Homework Question 1</u>

Prove that the set of all maps $\varphi : \mathbb{Z} \to \{0,1\}$ is uncountable.

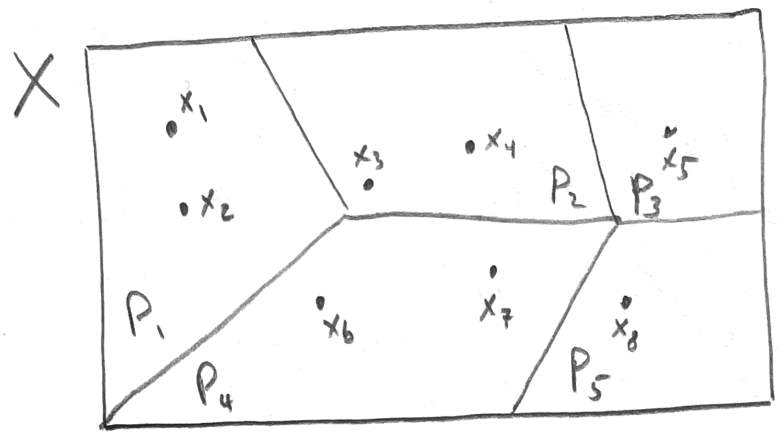
We now move to the second part of these notes:

Partitions + Equivalence Classes

It turns out that partitions and equivalence classes are the same thing, but it will take us a bit of time before we can show this. We will start with partitions, build some intuition, and then introduce the notion of an equivalence class.

Let X be a set. A partition of X may well be exactly what you suspect it is. It is a collection of subsets $\mathcal{P}$ such that the subsets are disjoint and cover X. (We will also require the subsets to be non-empty for technical reasons.) Each of the subsets $P \in \mathcal{P}$ we will call a part. Before giving the technical definition, let's look at an example of a partition of a finite set.

Let $X = \{x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8\}$.



$$\mathcal{P} = \{P_1, P_2, P_3, P_4, P_5\}$$ where

"part" $\longrightarrow P_1 = \{x_1, x_2\} \subset X$

"part" $\longrightarrow P_2 = \{x_3, x_4\} \subset X$

"part" $\longrightarrow P_3 = \{x_5\} \subset X$

"part" $\longrightarrow P_4 = \{x_6, x_7\} \subset X$

"part" $\longrightarrow P_5 = \{x_8\} \subset X$.

It turns out that $\mathcal{P}$ is a partition. Notice:

$$\mathcal{P} \subset \mathscr{P}(X)$$

and

$P_1 \in \mathcal{P}$, $P_2 \in \mathcal{P}$, etc., $P_1 \in \mathscr{P}(X)$, $P_2 \in \mathscr{P}(X)$, etc.

Now here is the formal definition.

DEF A subcollection $\mathcal{P} \subset \mathcal{S}(X)$ is a partition of X if it satisfies the following.

① If $P_1, P_2 \in \mathcal{P}$ and $P_1 \neq P_2$, then $P_1 \cap P_2 = \emptyset$.] disjoint

② If $P \in \mathcal{P}$, then $P \neq \emptyset$.] non-empty

③ $\bigcup_{P \in \mathcal{P}} P = X$.] $\mathcal{P}$ "covers" X

Any $P \in \mathcal{P}$ a partition, we call a part.

Partitions exist on infinite sets as well as on finite sets.

EXAMPLE

Consider the integers $\mathbb{Z}$. Let E denote the even integers and let O denote the odd integers.

Notice:

① $E \cap O = \emptyset$ (disjoint)

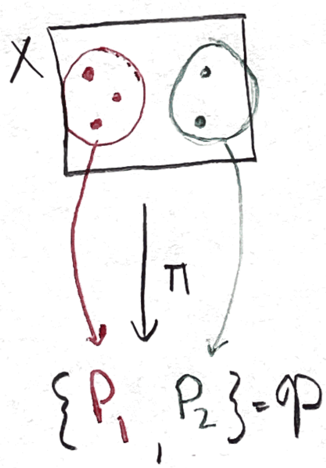
*Using our definitions of even and odd, we could prove (1), (2), and (3).

(2) $E \neq \emptyset$ and $O \neq \emptyset$ (non-empty)

(3) $E \cup O = \mathbb{Z}$ (covers $\mathbb{Z}$)

Thus, $\mathcal{P} = \{E, O\}$ is a partition. There are two parts: E and O.

<u>THM 1</u> For each partition $\mathcal{P}$ of a set X, define $\pi : X \longrightarrow \mathcal{P}$ as follows:



For $x \in X$ and $P \in \mathcal{P}$, we assign

$\boxed{\pi(x) = P \text{ if and only if } x \in P.}$

Then π is surjective.

<u>Sketch of proof:</u> First, observe that π is well-defined. Since $\mathcal{P}$ covers X, every x is in some part $P \in \mathcal{P}$, so π assigns a value (some $P \in \mathcal{P}$) to x, i.e. $\pi(x)$ is assigned for all $x \in X$.

Now, observe that π is indeed surjective. For any $P \in \mathcal{P}$, $P \neq \emptyset$, so there is some $x \in P$. Then $\pi(x) = P$.

$\boxtimes$

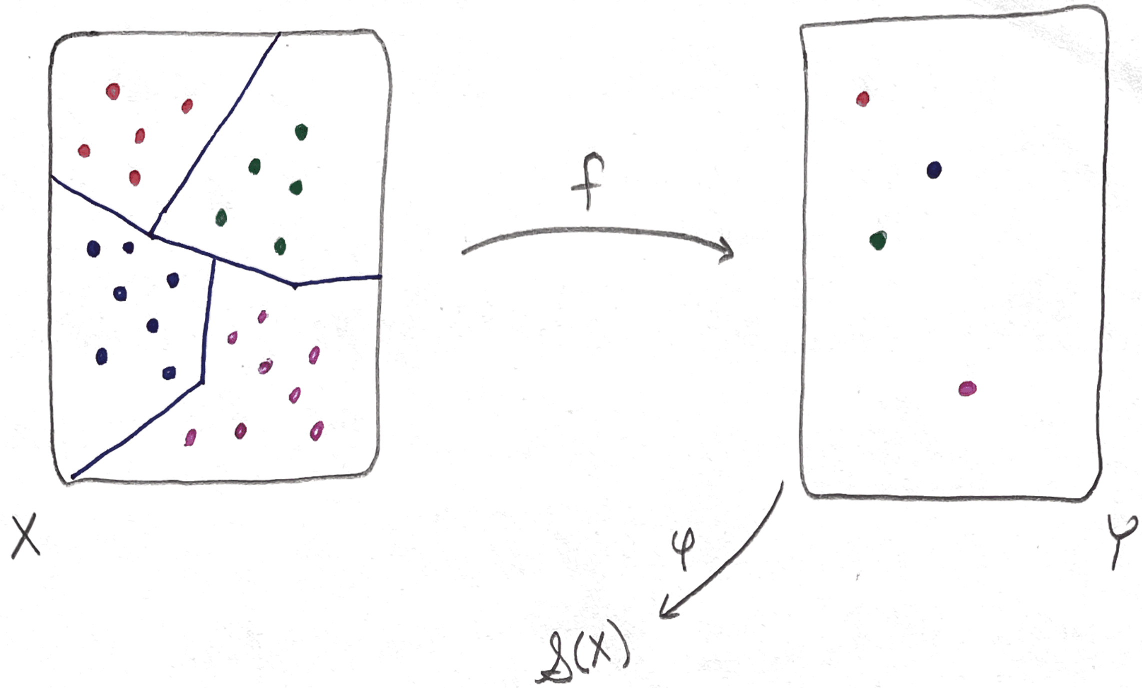
This theorem is telling us that given any partition, we can "project" the sets into the parts. The map π "puts each $x \in X$ into its corresponding part P," in some sense. You can also think of the map π as "blurring the elements of X," so that one can only see the parts, not the individual elements in each part. This leads us to a very natural question: can we generate partitions with maps? And we can...

<u>THM 2</u> For each surjective map $f: X \longrightarrow Y$, define $\varphi: Y \longrightarrow \mathcal{S}(X)$ by

$$\varphi(y) = f^{-1}(\{y\}) \left(= \{x \in X : f(x) = y\} \right)$$

for all $y \in Y$. Then φ is injective and $\varphi(Y)$ is a partition of X.

PIC (of THM 2)



Sketch of proof: First, observe that φ is injective.

① For any $y \in Y$, $\varphi(y) = f^{-1}(\{y\})$ is non-empty since f is surjective.

② For $y_1, y_2 \in Y$ such that $y_1 \neq y_2$, we have

$$f^{-1}(\{y_1\}) \cap f^{-1}(\{y_2\}) = \phi$$

meaning

$$\varphi(y_1) \cap \varphi(y_2) = \phi.$$

Since $\varphi(y_1)$ and $\varphi(y_2)$ are non-empty (by ①), we can conclude $\varphi(y_1) \neq \varphi(y_2)$.

Thus, φ is injective, as desired.

Next, observe $\varphi(Y)$ is a partition.

$\{* \varphi(y) \in \varphi(Y)\}$
part!

① For $P_1, P_2 \in \varphi(Y)$ such that $P_1 \neq P_2$,

disjoint → we have that $P_1 = \varphi(y_1)$ for some $y_1 \in Y$ and $P_2 = \varphi(y_2)$ for some other $y_2 \in Y$.

As before

$$\boxed{\varphi(y_1) \cap \varphi(y_2) = \phi.}$$

② Since f is surjective, $f^{-1}(\{y\}) \neq \phi$

non-empty → for every $y \in Y$. Thus $P_y = f^{-1}(\{y\})$ is non-empty.

③ Lastly, consider $\bigcup_{y \in Y} \varphi(y)$.

$$\bigcup_{y \in Y} \varphi(y) = \bigcup_{y \in Y} f^{-1}(\{y\})$$

covers →

$$= f^{-1}\left(\bigcup_{y \in Y} \{y\}\right),$$ since arbitrary unions play nicely with preimages!

$$= f^{-1}(Y)$$

$$= X,$$ as desired.

Hence, $\varphi(Y)$ is a partition.

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Homework Question 2 (don't actually need THM 1 or 2 to do this)

(a) Find the only partition of $\emptyset$.

(b) Find the only partition of $\{1\}$.

(c) Find both partitions of $\{1,2\}$.

(d) Find all five partitons of $\{1,2,3\}$.

(No need to prove these are partitions, or that none other exist.)

Great. Now we are going to leverage something that we have used a few times now, namely that for any surjective map $f: X \to Y$, the preimages of two different elements y_1 and y_2 are disjoint: $f^{-1}(\{y_1\}) \cap f^{-1}(\{y_2\}) = \emptyset$. This next theorem will be a bit tangential, but it will enable us to improve Cantor's Theorem from Week 7.

THM 3 Let X and Y be sets. Then

① there exists a surjective map $f: X \to Y$

if and only if

② there exists an injective map $g: Y \to X$.

Sketch of proof: First, assume ①. (We are proving ① $\Rightarrow$ 2).

Let $f: X \to Y$ be a surjective map.

Define $g: Y \to X$ as follows.

For each $y \in Y$, choose $x \in f^{-1}(\{y\})$.

(We can. f is surjective, so $f^{-1}(\{y\})$ is non-empty.) Now, assign

$g(y) =$ to this x !

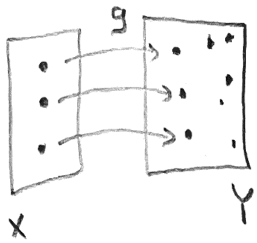
Now, notice, if $y_1 \neq y_2$, then

(*) $\longrightarrow$ $f^{-1}(\{y_1\}) \cap f^{-1}(\{y_2\}) = \emptyset$, which means $g(y_1)$ cannot equal $g(y_2)$ since they each live in disjoint sets.

Thus, $g(y_1) \neq g(y_2)$, and we can conclude that g is injective.

Now, assume ②. (We are proving ② $\Rightarrow$ ①.)

Let $g: Y \to X$ be injective. Then



for any $x \in g(Y)$, there is exactly one $y \in g^{-1}(\{x\})$.

Define $f(x) =$ this y! This will guarantee that once the map f is well-defined, it will be surjective. (We can rigorize this! $\{y\} = g^{-1}(\{x\})$, union to get all of Y.)

To ensure f is well-defined, we need to make sure we are assigning a value to $f(x)$ for all $x \in X$. Since g is not necessarily surjective, there may be $x \in X$ such that $x \notin g(Y)$.

For these x, let $f(x) = y$ for any $y \in Y$ (we don't care which $y \in Y$, we just need to make an assignment!). Then, f will be surjective, as desired.

$\boxed{/}$

Now, we can use Theorem 3 to upgrade Cantor's theorem.

THM4 (Upgraded Cantor's Theorem)

1. There is no surjective map from X to $\mathcal{P}(X)$.
2. There is no injective map from $\mathcal{P}(X)$ to X.
3. There is no bijective map from X to $\mathcal{P}(X)$.

Sketch of proof: 3 Was proven in Week 7.
1 Was proven in the homework for Week 7 (see answer key for homework for proof).
2 follows from 1 by applying THM3.

$\boxtimes$

Now, with that completed, we will turn back to Partitions and Equivalence Relations. We will now introduce the definition of an equivalence relation. When you read it, return to THM 1 of Week 7 and notice how this theorem actually proves that cardinal equivalence is really an equivalence relation! ... Well, modulo one issue...

DEF An <u>equivalence relation</u> on a set X is a subset $R \subset X \times X$ satisfying the following properties. (Note: $x \sim y$ below means $(x,y) \in R$.)

1. $x \sim x$ for each $x \in X$
2. If $x \sim y$, then $y \sim x$.
3. If $x \sim y$ and $y \sim z$, then $x \sim z$.

We read $x \sim y$ as x "is equivalent to" y.

(Extreme) EXAMPLES (of equivalence relations)

a) Say $x \sim y$ for all $x, y \in X$.

b) Say $x \sim y$ if and only if $x = y$.

Here is a general way to get an equivalence relation on a set.

... coming from a partition of a set.

THM 5 Let $\mathcal{P}$ be any partition of a set X. For $x, y \in X$, say $x \sim y$ if and only if x and y are elements in the same part $P \in \mathcal{P}$. Then "$\sim$" is an equivalence relation.

Sketch of proof: We must verify ①, ②, and ③ hold in the definition of an equivalence relation.

① For each $x \in X$, $x \in P$ for some $P \in \mathcal{P}$. ($\mathcal{P}$ covers X).
Then x is in same part as itself, so $x \sim x$.

② If $x \sim y$, then x is in the same part P as y.
Thus, y is in the same part P as x, so $y \sim x$.

③ If $x \sim y$ and $y \sim z$, then x, y, and z are all in the same part P, so x is in the same part as z. Thus $x \sim z$.

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Interestingly, we can make a partition out of an equivalence relation ...

THM 6 Let R be an equivalence relation on a set X.
For each $x \in X$, define

$$P_x := \{ y \in X : x \sim y \text{, i.e. } (x,y) \in R \}.$$

Then

(a) The collection $\mathcal{P} = \{ P_x : x \in X \}$ is a partition of X; and

(b) $x \sim y$ if and only if x and y belong to the same part of $\mathcal{P}$.

Sketch of proof: First we show a), that $\mathcal{P}$ is a partition.

a) Each P_x is non-empty since $x \in P_x$.] ← non-empty

Now, we will show that distinct P_x's are disjoint by showing that if P_x and P_y are not disjoint, then they are the same part. Suppose $P_x \cap P_y \neq \emptyset$. Let $z \in P_x \cap P_y$. Then $x \sim z$ and $y \sim z$, so $x \sim z$ and $z \sim y$, and we can conclude $x \sim y$, by the definition of an equivalence relation. This means that for all $z \in P_y$, $y \sim z$, so (since $x \sim y$), we have that $x \sim z$. Thus $z \in P_x$. By definition of subset, we see that $P_y \subset P_x$. A similar argument shows $P_x \subset P_y$. Thus, by an earlier theorem, we see that $P_x = P_y$.] disjoint

rem $\mathcal{P}$ is a set of sets, and we do not allow repeated elements in the set $\mathcal{P}$.

Lastly, since every $x \in X$ is in P_x,

$$\bigcup_{x \in X} P_x = X,$$

so we see that $\mathcal{P}$ covers X.] covers

Now we show b), which interprets our rule for the equivalence relation in terms of the partition.

If $x \sim y$, then we know $y \in P_x$, and by our argument above proving disjointness, we see that $P_x = P_y$. In other words, x and y belong to the same part.

Conversely, if x and y belong to the same part, then $P_x = P_y$, so $y \in P_x$ and we have that $x \sim y$.

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REMARK THM 5 tells us that any partition gives rise to an equivalence relation. THM 6 tells us that any equivalence relation gives rise to a partition. Combining these theorems, we see that a partition $\mathcal{P}$ on a set X is really the same thing! as an equivalence relation on X, where

$$\boxed{x \sim y \text{ if and only if } x \text{ and } y \text{ are in the same part of } \mathcal{P}.}$$

Homework Question 3

For each $m \in \mathbb{N}$, and $n_1, n_2 \in \mathbb{Z}$, we write

$$n_1 \equiv n_2 \mod m$$

(and say n_1 "is congruent to" n_2 "modulo" m)

if $n_1 - n_2 = qm$ for some $q \in \mathbb{Z}$. (In the case where $m = 2$, $n_1 \equiv n_2 \mod 2$ means that n_1 and n_2 are either both even or both odd.) Show that congruence modulo m is an equivalence relation on $\mathbb{Z}$ by proving that

1. $n \equiv n \mod m$ for all $n \in \mathbb{Z}$,
2. $n_1 \equiv n_2 \mod m$ implies $n_2 \equiv n_1 \mod m$, and
3. $n_1 \equiv n_2 \mod m$ and $n_2 \equiv n_3 \mod m$ implies $n_1 \equiv n_3 \mod m$.

Then, show that the <u>residue</u> <u>classes</u>, i.e. $\{(k+qm) : q \in \mathbb{Z}\}$ for $k = 0, 1, \ldots, m-1$, are the parts of the corresponding partition of $\mathbb{Z}$.