Lecture Notes: Week #7

Cardinal Equivalence and Diagonal Arguments

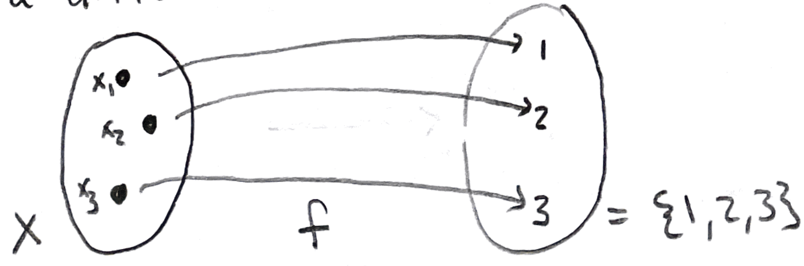
In this set of notes, we will largely return to the "definition, theorem, proof" style, except for when we are introducing a new proof technique. We will introduce the notion of cardinal equivalence, which we will come to understand as a way of counting that can be extended to counting an infinite number of objects. Of course, if cardinal equivalence only distinguished between different sizes of finite sets, and perhaps finite sets and infinite sets, it would not really tell us more than we already know: finite sets can be assigned a size (i.e. the number of elements) and infinite sets are ... well infinite. But, what we will discover is that the notion of cardinal equivalence can help us distinguish between different "sizes of infinity." So, without further ado, let's begin.

DEF Let X and Y be sets. If there is a bijection $f: X \to Y$, then we say that X and Y are cardinally equivalent, and write $X \sim Y$.

In other words, the existence of a bijection between two sets tells us the two sets are cardinally equivalent.

DEF If $X \sim \{1, 2, \dots, n\}$ for some $n \in \mathbb{N}$, then we write $|X| = n$, and say that X has size n. If $|X| = n$ for some $n \in \mathbb{N}$, then X is finite.

$\boxed{\text{Warning!!}}$ It might look like this is a definition that we already have, but it isn't. In a previous set of notes, we said $|X| = n$ just means X is a finite set with n elements in it. Here, we are explicitly defining it – i.e. we are defining how to count the number of elements in a set. In other words, this is what it means to count! You assign each element in the set exactly one number, and each element is assigned a different number. It is a bijective map!



THM 1 Let X, Y, and Z be sets.

① $X \sim X$, for each set X. (reflexive)

② If $X \sim Y$, then $Y \sim X$. (symmetric)

③ If $X \sim Y$ and $Y \sim Z$, then $X \sim Z$. (transitive)

REMARK The above theorem is telling us that cardinal equivalence is ① reflexive, ② symmetric, and ③ transitive. Next week, we will introduce the notion of an equivalence relation, and we will see that this theorem is telling us that cardinal equivalence is an equivalence relation.

REMARK As we will see next week, equivalence relations give us a way of "categorizing," for lack of a better term. Here, cardinal equivalence will enable us to categorize by cardinality. For finite sets, cardinality is the same as "size." In other words, for finite sets, cardinal equivalence categorizes by the size, or number of elements in the set. One way to think about this is as follows: if you view finite sets through the lens of cardinal equivalence, or "up to cardinal equivalence," the only thing you can see about a set is the number of elements in it. If you have two sets, say $X = \{1, 2, 3\}$, and $Y = \{$☺, 🌲, ✏$\}$, you cannot distinguish them ...

Cardinal equivalence makes them "look the same."

Sketch: (of proof of THM 1)

Let X, Y, and Z be sets.

1. Let $i_X : X \to X$ denote the identity function. In the last set of notes, we showed i_X is bijective. Hence, by the definition of cardinal equivalence, $X \sim X$ as desired.

2. Assume $X \sim Y$. Then there exists a bijective function $f : X \to Y$. Since f is bijective, there exists an inverse function f^{-1}, and by a homework problem, (#6, Week 5), we know f^{-1} is bijective. By the definition of the inverse function, $f^{-1} : Y \to X$. Since it is bijective, $Y \sim X$, by our definition of cardinal equivalence, as desired.

3. Assume $X \sim Y$ and $Y \sim Z$. Then there exist bijections $f : X \to Y$ and $g : Y \to Z$. By definition of the composite map, $g \circ f : X \to Z$. By THM 4 from Week 5, since f and g are both bijective, we can conclude $g \circ f$ is bijective. Hence, by definition of cardinal equivalence, $X \sim Z$. ∎

REMARK Notice that to prove cardinal equivalence of sets, we are proving the existence of a bijective map between the sets, which amounts to constructing a bijective map with given information.

Homework Question 1

Prove that for finite sets X and Y, $X \sim Y$ if and only if $|X| = |Y|$.

(You may assume LEMMA 1 without proving it. See below for Lemma.)

Tangent: Pigeonhole Principle.

Question Given 15 phones and 16 phone cases, can each phone get its own case?

(sure... we have enough cases.)

Given 16 phones and 15 cases, can each phone get its own case?

(nope... not enough cases....
one of the phones will be left out,
or we will have an awkward ordeal
while trying to fit two phones in a
single case ...)

The idea hiding behind these questions is one that we can formalize and use in our math proofs. This is something called the pigeonhole principle.

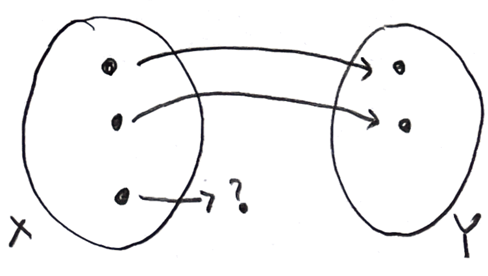
Pigeonhole Principle

If you have n objects and m containers for the objects, and $n > m$, then at least one container must hold more than one object.

We should be able to prove such a statement rigorously, but to do that, we need to rewrite the statement so that it fits nicely within our (current) theory of sets and counting.

LEMMA 1

Let $X = \{1, 2, \ldots, n\}$ and $Y = \{1, 2, \ldots, m\}$. If $n > m$, then there does not exist an injective map $f : X \longrightarrow Y$.



REMARK LEMMA 1 is logically equivalent to LEMMA $\tilde{1}$ below. (LEMMA $\tilde{1}$ is the contrapositive).

LEMMA $\tilde{1}$

Let $X = \{1, 2, \dots, n\}$ and $Y = \{1, 2, \dots, m\}$. If there exists an injective map $f: X \longrightarrow Y$, then $n \leq m$.

Time permitting, we will prove LEMMA 1 next week. But for the time being, will will return to our theoretical construction...

THM 2 (Cantor, 1891)

For no set X is $X \sim \mathcal{P}(X)$.

(i.e. There is no bijection from X to $\mathcal{P}(X)$, no matter what set X we pick!)

REMARK Notice that THM 2 applies to any set, finite or otherwise. However, we can prove this theorem with the machinery we have built so far if we assume X is finite:

Fact 1: If $|X| = n$, then $|\mathcal{P}(X)| = 2^n$. (THM 7, week 4)

Fact 2: $n < 2^n$ for all $n \in \mathbb{N}$. (Example, week 3)
(i.e., $n \neq 2^n$ for all $n \in \mathbb{N}$)

Combining facts 1 and 2 tell us that for any finite set such that $|X|=n$, we have $|X| \neq |\mathcal{S}(X)|$. Then, by Homework Question 1 in these notes, we can conclude X is not cardinally equivalent to $\mathcal{S}(X)$.

Very much relies on the pigeonhole principle!

(You can check the case of $X=\phi$ independently, or improve Fact 2 so as to include $n=0$.)

However, to prove this statement for a general set X, we need a new technique. In some sense, we can think of this new technique as a generalization of the pigeonhole principle, namely a way to find / construct an extra element (like having an extra object ... not enough containers). We call this style of argument a Diagonal Argument.

proof: (of THM 2)

Let X be a set. Assume for the sake of contradiction that there exists a bijection

$$f: X \longrightarrow \mathcal{S}(X).$$

Define a set A conditionally as follows:

$$A = \{ x \in X : x \notin f(x) \}.$$

This might look a bit odd, but it does make sense. Here, eack x is an element of X. The bijection f send x to an element of the powerset, i.e. $f(x) \in \mathcal{P}(X)$. The elements in the power set are subsets of X, so we can ask if x is in this particular subset. A is then the set of elements x such that $x \in f(x)$ where $f(x) \subset X$.

Now, observe: $A \subset X$. A contains only elements in X, and even if A is empty, $\emptyset \subset X$. This means no matter what A is, it is an element in the power set of X.

Since $f : X \longrightarrow \mathcal{P}(X)$ is a bijection, there is exactly one element $a \in X$ such that $f(a) = A$.

Now, we can ask a question.... is $a \in A$?

(Spend a moment to make sure this question makes sense!)

There are two possible scenarios: either $a \in A$ or $A \notin A$.

1 2

Okay, so...

① If $a \in A$, that means $a \notin f(a)$. $f(a) = A$, so $a \notin A$...

That ends in contradiction... so this case is not possible!

② If $a \notin A$, that means $a \in f(a)$. $f(a) = A$, so $a \in A$...

Also a contradiction! This case cannot occur either.

Since one of these two statements must be true, this is our contradiction! In other words, our original assumption was false: there is no bijection $f: X \to \mathcal{P}(X)$.

$\boxtimes$

Homework Question 2

Prove the following statement:

For no set X is there a surjective map $f: X \to \mathcal{P}(X)$.

(Hint: You can almost use the same proof as above, but you need to change a few key words.)

REMARK We can always find an injective map $f: X \to \mathcal{P}(X)$. Take $f(x) = \{x\}$.
This tells us that the problem with a bijective map $f: X \to \mathcal{P}(X)$ is that f cannot be surjective.

We will come back to this diagonal argument at the end of the notes.

DEF Let X be an infinite set. We say X is <u>countable</u> if $X \sim \mathbb{N}$, otherwise we say X is <u>uncountable</u>.

NOTE Some books will <u>also</u> call finite sets countable!

THM 3

1. $\mathbb{Z}$ is countable.
2. If X and Y are countable, so is $X \times Y$.
3. Each infinite subset of a countable set is countable.
4. $\mathbb{Q}$ is countable.

<u>Sketch</u>: We must exhibit bijections to $\mathbb{N}$ for each set above.

1. Consider $\mathbb{Z}$. We will construct a bijection $f: \mathbb{N} \longrightarrow \mathbb{Z}$, which will tell us $\mathbb{N} \sim \mathbb{Z}$. By THM 1 2, we can then conclude $\mathbb{Z} \sim \mathbb{N}$ as desired. It

turns out that there are many bijections we could construct. Here is one:

$$f(n) = \begin{cases} \frac{n}{2} & \text{if } n \text{ is even} \\ -\left(\frac{n-1}{2}\right) & \text{if } n \text{ is odd.} \end{cases}$$

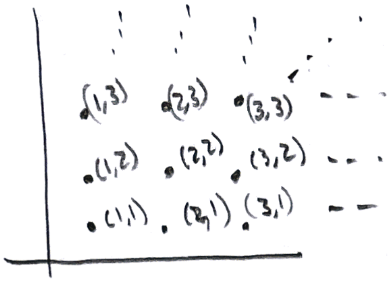
(To justify, we would need to show f is injective and surjective. This is not too hard, but it relies on techniques from the first few weeks.)

PIC

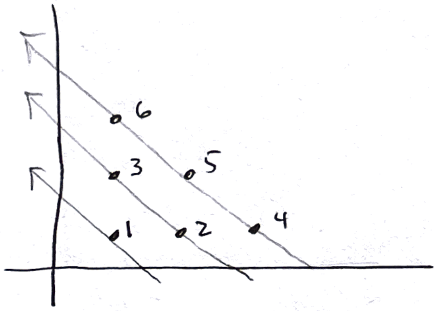
$$f: \begin{array}{ccc} 1 & \longrightarrow & 0 \\ 2 & \longrightarrow & 1 \\ 3 & \longrightarrow & -1 \\ 4 & \longrightarrow & 2 \\ 5 & \longrightarrow & -2 \\ 6 & \longrightarrow & 3 \\ 7 & \longrightarrow & -3 \\ \vdots & & \vdots \end{array}$$

(2) We will start with a simpler before doing the general case. Let $X = \mathbb{N}$ and $Y = \mathbb{N}$. (X and Y are countable since $X = \mathbb{N} \sim \mathbb{N}$ and $Y = \mathbb{N} \sim \mathbb{N}$, by reflectivity, i.e. THM 1 (1).)

We will construct a bijection $f: \mathbb{N} \times \mathbb{N} \longrightarrow \mathbb{N}$, Since $\mathbb{N} \times \mathbb{N}$ is the set of all ordered pairs:



we can get a bijection by listing these lattice points in the following order



i.e. $f(1,1) = 1$, $f(2,1) = 2$, $f(1,2) = 3$, etc.

Fortunately, we can write this map down concretely.

$$f(n,m) = \frac{1}{2}(n+m-2)(n+m-1) + n.$$

Now, we just need to show f is both injective and surjective. Also not too hard to check... but surjectivity leads to an interesting equation.

Provided this claim is true, we have shown that $\mathbb{N} \times \mathbb{N} \sim \mathbb{N}$. Now we have to do this for countable X and countable Y.

Let X and Y be countable sets. This means that there exists bijections $f_X : X \to \mathbb{N}$ and $f_Y : Y \to \mathbb{N}$. Let $f : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ be a bijection (which we know exists from the first case). Now, let

$g : X \times Y \to \mathbb{N}$ be the map defined by

$$g(x,y) = f(f_X(x), f_Y(y)).$$

Homework Question 3

Show that the map g above is a bijection.

Assuming the homework problem is true, this completes the proof: $X \times Y \sim \mathbb{N}$, as desired.

③ Let A be a countable subset of $X = \mathbb{N}$. We are again starting with a simpler case, we will let X be an arbitrary countable set after we prove this for $X = \mathbb{N}$.

We will construct a map $f : \mathbb{N} \to A$, that is bijective, and its inverse will give us $A \sim \mathbb{N}$.

We will need a few facts to do it.

Fact 1: Every element in A is distinct.

This is immediate from our definition of sets.

Fact 2: Any two elements in A can be ordered, as in for any $x, y \in A$, $x < y$ or $y < x$. This is immediate from the trichotomy axiom at the begining of class (axioms of integers), and Fact 1.

Fact 3 A has a minimal element. This follows from an argument similar to the one we gave at the end of Week 3 (see the video!! lecture) to show that a finite set has a maximal element. Since $1 \in \mathbb{N}$ is minimal (there are no $n \in \mathbb{N}$ such that $n < 1$), we can use this to show A will inherit the property of having a minimal element. (Non-rigorous version: Let $n \in A$. It exists since A is countable. Then there are at most $(n-1)$ elements less than n.)

Warning! This is basically the Well-Ordering principle.

Combining facts 1, 2, and 3 tells us we can construct a function using induction.

<u>Base Case</u>: Let n_1 be the minimal element in A. Assign $f(1) = n_1$.

<u>Inductive Step</u>: Assume $f(k)$ is assigned for some $k \geq 1$. Let n_{k+1} be the least element of $\{a \in A : a > f(k)\}$.

inductive hypothesis tells you this exists!

Assign $f(k+1) = n_{k+1}$.

Now, notice, with this function, $f(k) < f(k+1)$ for all k, so the function is injective. The function will also be surjective: $f(n) \geq n$, which we can show by induction. This means for any $m \in A$, we assigned $f(m)$, and either $f(m)$ hits m $(f(m)=m)$ or $f(m) > m$, in which case there is some $k < m$ such that $f(k) = m$.

Thus, f is a bijection, and $A \sim \mathbb{N}$.

Now, we must deal with an arbitrary countable

X and an infinite subset $A \subset X$. (i.e. we need to consider the case where $X \neq \mathbb{N}$.)

Since X is countable, there exists a bijection $f_X : X \to \mathbb{N}$. We need an order on X, and we can inherit one from $\mathbb{N}$. Let $x_1, x_2 \in X$ and say $x_1 < x_2$ if $f_X(x_1) < f_X(x_2)$. In fact, since $A \subset X$, we inherit the order on A as well: for $a_1, a_2 \in A$, we say $a_1 < a_2$ if $f(a_1) < f(a_2)$.

Now the proof can proceed as in the previous case. Define $f : \mathbb{N} \to A$ inductively using the order, then $f^{-1} : A \to \mathbb{N}$ is the desired bijection telling us $A \sim \mathbb{N}$.

(4) Lastly, to show that $\mathbb{Q}$ is countable, we should recall that any rational number can be written uniquely as a reduced fraction, $q = \frac{m}{n}$, where the greatest common denominator

of m and n is 1 ($\frac{3}{2}$ is reduced, but $\frac{6}{4}$ is not), $m \in \mathbb{Z}$, and $n \in \mathbb{N}$.

This means there is a map $f: \mathbb{Q} \longrightarrow \mathbb{Z} \times \mathbb{N}$ given by $f(\frac{m}{n}) = (m,n)$, provided $\frac{m}{n}$ is reduced.

The map is injective: if $f(\frac{m_1}{n_1}) = f(\frac{m_2}{n_2})$, then $(m_1, n_1) = (m_2, n_2)$, hence $\frac{m_1}{n_1} = \frac{m_2}{n_2}$ as desired.

However, the map is <u>not</u> surjective. $\frac{6}{4}$ is not reduced, so $f(\frac{6}{4})$ is not defined (meaning we miss $(6,4)$).

But this is not a problem. Notice, if we restict f to its image, i.e. $f(\mathbb{Q})$, then the restriction becomes bijective: Define

$$g: \mathbb{Q} \longrightarrow f(\mathbb{Q})$$

where $g(\frac{m}{n}) = f(\frac{m}{n})$. Now, g is bijective.

This is good because we have a bijective map from $\mathbb{Q}$ to a countable set — by ①, we know that $\mathbb{Z}$ is countable, by reflection we know that $\mathbb{N}$ is countable (i.e. $\mathbb{N} \sim \mathbb{N}$), and by ② we know that $\mathbb{Z} \times \mathbb{N}$ is countable.

We also need to observe that f(Q) is an infinite subset of ZxN, a countable set. Then by (3), we see that f(Q) is a countable set.

Let $h:$ f(Q) [redacted] $\longrightarrow \mathbb{N}$ be a bijective map (which exists since $\mathbb{Z} \times \mathbb{N} \sim \mathbb{N}$), and take the composition map $h \circ g : \mathbb{Q} \longrightarrow \mathbb{N}$. Since h and g are bijective, by THM 4 from Week 5, we know $h \circ g$ is bijective, hence $\mathbb{Q} \sim \mathbb{N}$ as desired! $\mathbb{Q}$ is countable!

$\square$

Next, we will show that $\mathbb{R}$ is not countable. In other words, $\mathbb{R}$ is uncountable. We will need a little Lemma to show this, and we will prove this at the begining of next week's lecture notes.

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Homework Question 4

Let A be an infinite subset of a countable set.

Show that $\mathcal{S}(A \times \mathbb{Z})$ is uncountable.