Homework #8 Answer Key

1. Let $S = \{ \varphi \mid \varphi : \mathbb{Z} \to \{0,1\} \}$ be a set of maps. We will show that this set is uncountable. Assume for the sake of contradiction that S is countable. Then, $S \sim \mathbb{N}$, and by symmetry, $\mathbb{N} \sim S$. Thus, there is a bijection $f : \mathbb{N} \to S$. We can use this bijection to label our maps in S:

"enumerate"

$$f(1) = \varphi_1 \quad \text{label}$$
$$f(2) = \varphi_2$$
$$\vdots$$
$$f(k) = \varphi_k .$$
$$\vdots$$

Now, notice that $\mathbb{Z}$ is countable, by THM 3 from Week 7. Thus, $\mathbb{Z} \sim \mathbb{N}$, and by symmetry, $\mathbb{N} \sim \mathbb{Z}$. This means there exists a bijection $g : \mathbb{N} \to \mathbb{Z}$. We can use this bijection to label our integers:

$$g(1) = n_1$$
$$g(2) = n_2$$
$$\vdots$$
$$g(k) = n_k .$$
$$\vdots$$

Now, take a map in S, say φ_k, and notice that φ_k assigns a 0 or a 1 for every integer. This means

$$\begin{array}{ccccc} \mathbb{Z}: & n_1 & n_2 & n_3 & n_4 \\ & \big\downarrow \varphi_k & \big\downarrow \varphi_k & \big\downarrow \varphi_k & \big\downarrow \\ & 0 \text{ or } 1 & 0 \text{ or } 1 & 0 \text{ or } 1 & 0 \text{ or } 1. \end{array}$$

We can list the maps in S and record the values for each integer:

$$\begin{array}{c|ccccc} & n_1 & n_2 & n_3 & n_4 & \cdots \\ \hline \varphi_1 & b_{11} & b_{12} & b_{13} & b_{14} & \cdots \\ \varphi_2 & b_{21} & b_{22} & b_{23} & b_{24} & \cdots \\ \varphi_3 & b_{31} & b_{32} & b_{33} & b_{34} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \\ \varphi_k & b_{k1} & b_{k2} & b_{k3} & b_{k4} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \end{array}$$

Here, b_{ij} is the value of $\varphi_i(n_j)$. In other words:

$$\boxed{\varphi_i(n_j) = b_{ij}}$$ and $b_{ij} \in \{0,1\}$

(Notice, we have identified each map with a one-sided string of 0's and 1's.)

Now, define a new map φ^* where for every $i \in \mathbb{N}$,

$$\varphi^*(n_i) \neq \varphi_i(n_i).$$

We can do this since $\varphi_i(n_i) \in \{0,1\}$, so we can always choose the number that $\varphi_i(n_i)$ is not. (For example, if $\varphi_i(n_i) = 1$, then we pick $\varphi^*(n_i) = 0$.)

Now, notice that φ^* is a map from $\mathbb{Z}$ to $\{0,1\}$, so it should be in our list. In other words, $\varphi^* = \varphi_k$ for some $k \in \mathbb{N}$. However, $\varphi^*(n_k) \neq \varphi_k(n_k)$ for any k, so it cannot be in our list! Thus, our assertion that S is countable is wrong. We can conclude S is uncountable. (Note that there are infinitely many maps in S.)

2. a) $\mathcal{P} = \emptyset$

b) $\mathcal{P} = \{\{1\}\}$

c) $\mathcal{P}_1 = \{\{1\}, \{2\}\}$

$\mathcal{P}_2 = \{\{1,2\}\}$

d) $\mathcal{P}_1 = \{\{1\}, \{2\}, \{3\}\}$

$\mathcal{P}_2 = \{\{1,2\}, \{3\}\}$

$\mathcal{P}_3 = \{\{1,3\}, \{2\}\}$

$\mathcal{P}_4 = \{\{2,3\}, \{1\}\}$

$\mathcal{P}_5 = \{\{1,2,3\}\}$.

3.

see next page!

3. We first show that "congruence modulo m" is an equivalence relation by showing that it satisfies the definition of an equivalence relation. Let $n_1, n_2, n_3, n \in \mathbb{Z}$.

① $n - n = 0 = 0 \cdot m$.

Since $0 \in \mathbb{Z}$, $n \equiv n \mod m$ as desired.

② Assume $n_1 \equiv n_2 \mod m$. Then $n_1 - n_2 = qm$ for some $q \in \mathbb{Z}$. That means that $n_2 - n_1 = (-q)m$, and $-q \in \mathbb{Z}$. Thus, $n_2 \equiv n_1 \mod m$, as desired.

③ Assume $n_1 \equiv n_2 \mod m$ and $n_2 \equiv n_3 \mod m$. Then $n_1 - n_2 = q_1 m$ and $n_2 - n_3 = q_2 m$ for some integers q_1 and q_2. Notice that:

$$(n_1 - n_2) + (n_2 - n_3) = (q_1 m) + (q_2 m)$$

$$n_1 - n_3 = (q_1 + q_2) m.$$

Since $q_1 + q_2 \in \mathbb{Z}$, $n_1 \equiv n_3 \mod m$, as desired.

Thus, "congruence modulo m" satisfies the definition of an equivalence relation, so it is an equivalence relation.

We now show that the residue classes are the parts of the corresponding partition.

By Theorem 6, we know all of the parts are of the form

$$P_n = \{ \tilde{n} \in \mathbb{Z} : n \sim \tilde{n} \}$$

for some $n \in \mathbb{Z}$. Notice

$$\begin{aligned} P_n &= \{ \tilde{n} \in \mathbb{Z} : n \sim \tilde{n} \} \\ &= \{ \tilde{n} \in \mathbb{Z} : n \equiv \tilde{n} \bmod m \} \\ &= \{ \tilde{n} \in \mathbb{Z} : n - \tilde{n} = qm \text{ for some } q \in \mathbb{Z} \} \end{aligned}$$

Using the Division Algorithm, we know that

$$\begin{cases} n = q_1 m + k \\ \tilde{n} = q_2 m + \tilde{k} \end{cases}$$

for some integers q_1, q_2, k, and $\tilde{k}$ where $0 \leq k \leq m-1$ and $0 \leq \tilde{k} \leq m-1$. So,

$$P_{n = q_1 m + k} = \{ q_2 m + \tilde{k} \in \mathbb{Z} : q_2 \in \mathbb{Z},\ 0 \leq \tilde{k} \leq m-1, \text{ and } (q_1 m + k) - (q_2 m + \tilde{k}) = qm \}$$

$$= \{ q_2 m + \tilde{k} \in \mathbb{Z} : q_2 \in \mathbb{Z},\ 0 \le \tilde{k} \le m-1, \text{ and } k - \tilde{k} = (q_2 - q_1 + q)m \}.$$

Now, observe that $|k - \tilde{k}| \le m-1$ since $0 \le k \le m-1$ and $0 \le \tilde{k} \le m-1$. Thus, $|(q_2 - q_1 + q)m| \le m-1$, and since $m \in \mathbb{N}$, $|q_2 - q_1 + q| \le \frac{m-1}{m}$. $\frac{m-1}{m} < 1$ for all $m \in \mathbb{N}$, since $m-1 < m$, so we see that since $q_2 - q_1 + q$ must be an integer, the only valid solution is $q_2 - q_1 + q = 0$.

Hence,

$$P_{n = q_1 m + k} = \{ q_2 m + \tilde{k} \in \mathbb{Z} : q_2 \in \mathbb{Z},\ 0 \le \tilde{k} \le m-1, \text{ and } k - \tilde{k} = 0 \}$$
$$= \{ q_2 m + k \in \mathbb{Z} : q_2 = \mathbb{Z} \}$$

This is the same as

$$\{ qm + k \in \mathbb{Z} : q \in \mathbb{Z} \}$$

for some $0 \le k \le m-1$. Now, since n was arbitrary in the beginning, we see that any part must be of this form for some k where $0 \le k \le m-1$.

We now need to show that every k occurs as a part, i.e. $\{qm+k : q \in \mathbb{Z}\}$ is non-empty for each $k \in \{0, 1, \ldots, m-1\}$. To see this, notice $0 \in \mathbb{Z}$ and

$$0 \in \{qm+0 : q \in \mathbb{Z}\} \quad \text{since } 0 = 0 \cdot m + 0,$$

$$1 \in \{qm+1 : q \in \mathbb{Z}\} \quad \text{since } 1 = 0 \cdot m + 1,$$

$$\vdots$$

$$m-1 \in \{qm+(m-1) : q \in \mathbb{Z}\} \quad \text{since } m-1 = 0 \cdot m + 1.$$

Thus, each set is non-empty, so it is really a part!

$\blacksquare$

(Notice, we get the partition for free by Theorem 6!)