# Homework #7 Answer Key

1. <u>proof</u>: First, assume $|X| \neq |Y|$. (We are proving the contrapositive of one of the directions of the implication.) Let $|X| = n$ and let $|Y| = m$. Then either $m < n$ or $n < m$

<u>Case 1</u>: $m < n$.

By <u>Lemma 1</u> (the pigeonhole principle), there does not exist an injective map $f: X \longrightarrow Y$.

Thus, there cannot exist a bijective map $f: X \longrightarrow Y$ since bijective maps must be injective, and we can conclude $X \not\sim Y$, as desired.

<u>Case 2</u>: $n < m$

By <u>Lemma 1</u>, there does not exist an injective map $g: Y \longrightarrow X$. As in Case 1, this means there cannot be a bijective map $g: Y \longrightarrow X$, so we see that $Y \not\sim X$. By THM 1 ②, we can conclude $X \not\sim Y$ (contrapositive), as desired.

Second, assume $|X| = |Y|$. (We will prove the second direction directly.) Let $|X| = n = |Y|$.

By the definition of size, this means there exist bijections $f_X: X \longrightarrow \{1, 2, \ldots, n\}$ and

$f_Y : Y \longrightarrow \{1, 2, \ldots, n\}$. Notice, $f_Y^{-1}$ exists since $f_Y$ is a bijection, and furthermore, $f_Y^{-1}$ is a bijection (by an old homework problem).

$$\begin{tikzcd}
X \arrow[rr, dashed, "f_Y^{-1} \circ f_X"] \arrow[dr, "f_X"'] & & Y \\
& \{1, 2, \ldots, n\} \arrow[ur, "f_Y^{-1}"'] &
\end{tikzcd}$$

Then, by definition of the composite map, $f_Y^{-1} \circ f_X : X \longrightarrow Y$ and by THM 4 from Week #5, since $f_Y^{-1}$ and $f_X$ are bijections, we have that $f_Y^{-1} \circ f_X$ is a bijection.

Thus, $X \sim Y$, as desired. $\blacksquare$

2. <u>proof:</u> Let $X$ be a set and assume for the sake of contradiction that there exists a surjection

$$f : X \longrightarrow \mathcal{P}(X).$$

Define a set $A$ as follows:

$$A = \{x \in X : x \notin f(x)\}.$$

Now, observe: $A \subset X$ (even if $A = \emptyset$), so $A \in \mathcal{S}(X)$. Since $f: X \longrightarrow \mathcal{S}(X)$ is surjective, there exists some $a \in X$ such that $f(a) = A$.

Now observe the following:

① If $a \in A$, then $a \in f(a)$, and by definition of the set $A$, we conclude $a \notin A$. C'est ridicule!

② If $a \notin A$, then $a \notin f(a)$, and by definition of the set $A$, we conclude $a \in A$. C'est ridicule!

Since both ① and ② lead to a contradiction, and either ① or ② must be true, we see this is a contradiction, hence our assumption that there exists a surjective map must be incorrect. Thus, there is no surjective map $f: X \longrightarrow \mathcal{S}(X)$, as desired. $\blacksquare$

3. <u>proof</u>: Let $g: X \times Y \to \mathbb{N}$ be defined as

$g(x,y) = f(f_X(x), f_Y(y))$ where $f: \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ is a bijection, $f_X: X \to \mathbb{N}$ is a bijection, and $f_Y: Y \to \mathbb{N}$ is a bijection.

<u>$g$ is injective</u>: Let $(x_1, y_1), (x_2, y_2) \in \mathbb{N}$.

Assume $g(x_1, y_1) = g(x_2, y_2)$. Then $f(f_X(x_1), f_Y(y_1)) = f(f_X(x_2), f_Y(y_2))$. Since $f$ is injective, $(f_X(x_1), f_Y(y_1)) = (f_X(x_2), f_Y(y_2))$, meaning both $f_X(x_1) = f_X(x_2)$ and $f_Y(y_1) = f_Y(y_2)$. By injectivity of $f_X$, we can conclude $x_1 = x_2$. By injectivity of $f_Y$, we can conclude $y_1 = y_2$. Thus, $(x_1, y_1) = (x_2, y_2)$, and we can conclude that $g$ is injective.

<u>$g$ is surjective</u>: Let $n \in \mathbb{N}$ be any natural number. We need to show that there is at least one $(x,y) \in X \times Y$ such that $g(x,y) = n$. Since $f: \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ is surjective, we know that there exists some $(n_1, n_2) \in \mathbb{N} \times \mathbb{N}$ such that

$f(n_1, n_2) = n$. Since $f_X$ is surjective, there is some $x$ such that $f_X(x) = n_1$. Similarly, since $f_Y$ is surjective, there exists some $y$ such that $f_Y(y) = n_2$.

Then

$$
\begin{aligned}
g(x,y) &= f(f_X(x), f_Y(y)) \\
&= f(n_1, n_2) \\
&= n,
\end{aligned}
$$

so we see that $g$ is surjective, as desired. $\blacksquare$

4. proof: Assume for the sake of contradiction that $\mathcal{S}(A \times \mathbb{Z})$ is countable. Then $\mathcal{S}(A \times \mathbb{Z}) \sim \mathbb{N}$, so there exists a bijection $f: \mathcal{S}(A \times \mathbb{Z}) \longrightarrow \mathbb{N}$.

Now, observe that by THM3 ③, $A$ is countable. By THM3 ①, $\mathbb{Z}$ is countable. By THM3 ②, $A \times \mathbb{Z}$ is countable. Thus, $A \times \mathbb{Z} \sim \mathbb{N}$ and there exists a bijection $g: A \times \mathbb{Z} \longrightarrow \mathbb{N}$.

Now, notice the following.

$$\begin{tikzcd}
\mathcal{S}(A \times \mathbb{Z}) \arrow[dr, "f"'] & & A \times \mathbb{Z} \arrow[ll, dashed, "f^{-1} \circ g"'] \arrow[dl, "g"] \\
& \mathbb{N} &
\end{tikzcd}$$

Since $f$ is bijective, $f^{-1}$ exist and is bijective. Then, by definition of the composite map, $f^{-1} \circ g : A \times \mathbb{Z} \to \mathcal{S}(A \times \mathbb{Z})$, and by THM 4 from Week 5, $f^{-1} \circ g$ is bijective since both $f^{-1}$ and $g$ are bijective. This means that $A \times \mathbb{Z} \sim \mathcal{S}(A \times \mathbb{Z})$, but this contradicts Cantor's Theorem (THM2). Thus, $\mathcal{S}(A \times \mathbb{Z})$ is uncountable.

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