

Homework #7 Answer Key

1. proof: First, assume $|X| \neq |Y|$. (We are proving the contrapositive of one of the directions of the implication.) Let $|X| = n$ and let $|Y| = m$. Then either $m < n$ or $n < m$.

Case 1: $m < n$.

By Lemma 1 (the pigeonhole principle), there does not exist an injective map $f: X \rightarrow Y$.

Thus, there cannot exist a bijective map $f: X \rightarrow Y$ since bijective maps must be injective, and we can conclude $X \not\cong Y$, as desired.

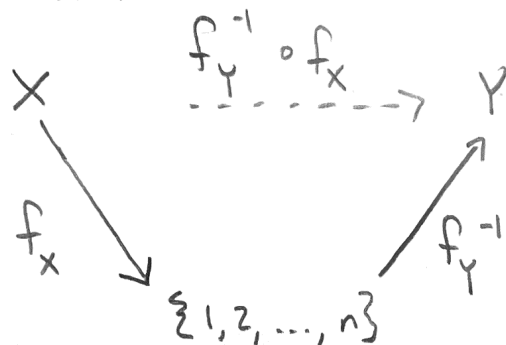
Case 2: $n < m$

By Lemma 1, there does not exist an injective map $g: Y \rightarrow X$. As in Case 1, this means there cannot be a bijective map $g: Y \rightarrow X$, so we see that $Y \not\cong X$. By THM 1(2), we can conclude $X \not\cong Y$ (contrapositive), as desired.

Second, assume $|X| = |Y|$. (We will prove the second direction directly.) Let $|X| = n = |Y|$.

By the definition of size, this means there exist bijections $f: X \rightarrow \{1, 2, \dots, n\}$ and

$f_Y : Y \rightarrow \{1, 2, \dots, n\}$. Notice, f_Y^{-1} exists since f_Y is a bijection, and furthermore, f_Y^{-1} is a bijection (by an old homework problem).



Then, by definition of the composite map, $f_Y^{-1} \circ f_X : X \rightarrow Y$ and by THM 4 from Week #5, since f_Y^{-1} and f_X are bijections, we have that $f_Y^{-1} \circ f_X$ is a bijection.

Thus, $X \sim Y$, as desired.



2. proof: Let X be a set and assume for the sake of contradiction that there exists a surjection

$$f : X \rightarrow \mathcal{S}(X).$$

Define a set A as follows:

$$A = \{x \in X : x \notin f(x)\}.$$

Now, observe: $A \subset X$ (even if $A = \emptyset$), so $A \in \mathcal{S}(X)$.

Since $f: X \rightarrow \mathcal{S}(X)$ is surjective, there exists some $a \in X$ such that $f(a) = A$.

Now observe the following:

① If $a \in A$, then $a \in f(a)$, and by definition of the set A , we conclude $a \notin A$. C'est ridicule!

② If $a \notin A$, then $a \notin f(a)$, and by definition of the set A , we conclude $a \in A$. C'est ridicule!

Since both ① and ② lead to a contradiction, and either ① or ② must be true, we see this is a contradiction, hence our assumption that there exists a surjective map must be incorrect. Thus, there is no surjective map $f: X \rightarrow \mathcal{S}(X)$, as desired.

▣

3.

proof:Let $g: X \times Y \rightarrow \mathbb{N}$ be defined as

$$g(x, y) = f(f_X(x), f_Y(y)) \quad \text{where } f: \mathbb{N} \times \mathbb{N} \rightarrow \mathbb{N}$$

is a bijection, $f_X: X \rightarrow \mathbb{N}$ is a bijection, and $f_Y: Y \rightarrow \mathbb{N}$ is a bijection. g is injective: Let $(x_1, y_1), (x_2, y_2) \in \mathbb{N}$.Assume $g(x_1, y_1) = g(x_2, y_2)$. Then

$$f(f_X(x_1), f_Y(y_1)) = f(f_X(x_2), f_Y(y_2)). \quad \text{Since}$$

$$f \text{ is injective, } (f_X(x_1), f_Y(y_1)) = (f_X(x_2), f_Y(y_2)),$$

meaning both $f_X(x_1) = f_X(x_2)$ and $f_Y(y_1) = f_Y(y_2)$.By injectivity of f_X , we can conclude $x_1 = x_2$. Byinjectivity of f_Y , we can conclude $y_1 = y_2$. Thus,

$$(x_1, y_1) = (x_2, y_2), \quad \text{and we can conclude that}$$

 g is injective. g is surjective: Let $n \in \mathbb{N}$ be any natural

number. We need to show that there is at

least one $(x, y) \in X \times Y$ such that $g(x, y) = n$.Since $f: \mathbb{N} \times \mathbb{N} \rightarrow \mathbb{N}$ is surjective, we knowthat there exists some $(n_1, n_2) \in \mathbb{N} \times \mathbb{N}$ such that

$f(n_1, n_2) = n$. Since f_x is surjective, there is some x such that $f_x(x) = n_1$. Similarly, since f_y is surjective, there exists some y such that $f_y(y) = n_2$. Then

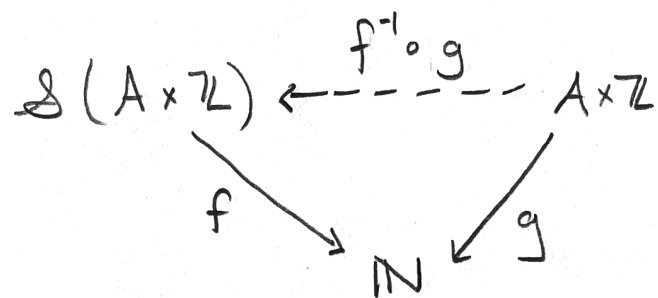
$$\begin{aligned}g(x, y) &= f(f_x(x), f_y(y)) \\ &= f(n_1, n_2) \\ &= n,\end{aligned}$$

so we see that g is surjective, as desired. \square

4. proof: Assume for the sake of contradiction that $\mathcal{S}(A \times \mathbb{Z})$ is countable. Then $\mathcal{S}(A \times \mathbb{Z}) \sim \mathbb{N}$, so there exists a bijection $f: \mathcal{S}(A \times \mathbb{Z}) \rightarrow \mathbb{N}$.

Now, observe that by THM 3 ③, A is countable. By THM 3 ①, \mathbb{Z} is countable. By THM 3 ②, $A \times \mathbb{Z}$ is countable. Thus, $A \times \mathbb{Z} \sim \mathbb{N}$ and there exists a bijection $g: A \times \mathbb{Z} \rightarrow \mathbb{N}$.

Now, notice the following.



Since f is bijective, f^{-1} exist and is bijective. Then, by definition of the composite map, $f^{-1} \circ g: A \times \mathbb{Z} \rightarrow \mathcal{P}(A \times \mathbb{Z})$, and by THM 4 from Week 5, $f^{-1} \circ g$ is bijective since both f^{-1} and g are bijective. This means that $A \times \mathbb{Z} \sim \mathcal{P}(A \times \mathbb{Z})$, but this contradicts Cantor's Theorem (THM2). Thus, $\mathcal{P}(A \times \mathbb{Z})$ is uncountable.

□