Homework #5 Answer Key

1. proof: Assume A, B, and C are finite sets. Then

$$|A \cup B \cup C| = |(A \cup B) \cup C| \text{ by associativity,}$$

$$= |(A \cup B)| + |C| - |(A \cup B) \cap C| \text{ by THM 8,}$$

$$= |A| + |B| - |A \cap B| + |C| - |(A \cup B) \cap C| \text{ by THM 8,}$$

$$= |A| + |B| + |C| - |A \cap B| - |(A \cap C) \cup (B \cap C)| \text{ by distributivity,}$$

$$= |A| + |B| + |C| - |A \cap B| - \left(|A \cap C| + |B \cap C| - |(A \cap C) \cap (B \cap C)|\right),$$

by THM 8,

$$= |A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C| + |(A \cap C) \cap (C \cap B)|$$

by commutativity of the intersection operation,

$$= |A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C| + |(A \cap (C \cap C)) \cap B|,$$

by associativity of the intersection operation.

To finish the proof, we need to show that $C \cap C = C$, then apply commutativity one last time. (Note that this could have been avoided!)

Let $x \in C$. Then $x \in C$ if and only if $x \in C$ and $x \in C$, which is true if and only if $x \in C \cap C$, by definition of the intersection. Thus, by defintion of set equality, $C = C \cap C$, as desired.

Continuing the computation, we see:

$$|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C| + |(A \cap C) \cap B|$$

since $C = C \cap C$,

$$= |A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C| + |A \cap (C \cap B)|$$

by associativity,

$$= |A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C| + |A \cap B \cap C|$$

by commutativity,

which completes the proof.

$\boxtimes$

The analogous theorem for $|A \cup B \cup C \cup D|$:

THM

Let $A, B, C,$ and D be finite sets. Then

$$\begin{aligned}|A \cup B \cup C \cup D| = {} & |A| + |B| + |C| + |D| - |A \cap B| - |A \cap C| - |A \cap D| - |B \cap C| \\ & - |B \cap D| - |C \cap D| + |A \cap B \cap C| + |A \cap B \cap D| + |A \cap C \cap D| \\ & + |B \cap C \cap D| - |A \cap B \cap C \cap D|.\end{aligned}$$

Note: this is something called the inclusion-exclusion principle.

2. <u>DEF</u> For arbitrary sets X, Y, and Z, we define the <u>Cartesian product</u> by

$$X \times Y \times Z = \{(x,y,z) : x \in X, y \in Y, z \in Z\}.$$

Here, (x,y,z) is an ordered pair.

<u>THM</u> If X, Y, and Z are finite sets, then

$$|X \times Y \times Z| = |X||Y||Z|.$$

3. <u>proof:</u> (We will prove a set equality for a Cartesian product.)

Assume A, B, and C are any sets such that B and C are disjoint. We will show that $A \times (B \sqcup C) = (A \times B) \sqcup (A \times C)$, and along the way, show that $(A \times B) \sqcup (A \times C)$ is well-defined (i.e. really is a disjoint union).

Let (a,b) be an ordered pair in $A \times (B \sqcup C)$.

$(a,b) \in A \times (B \sqcup C) \iff a \in A$ and $b \in B \sqcup C$, by definition of the cartesian product,

$\iff a \in A$ and $b \in B \cup C$, where $B \cap C = \emptyset$, by definition of the disjoint union,

$\iff a \in A$ and $(b \in B$ or $b \in C)$, where $B \cap C = \emptyset$, by definition of the union,

$\iff a \in A$ and $(b \in B$ or $b \in C)$, but not both, since there is no $x \in B \cap C$.

$\iff (a \in A$ and $b \in B)$ or $(a \in A$ and $b \in C)$ but not both.

$\iff (a,b) \in A \times B$ or $(a,b) \in A \times C$, but not both.

$\iff (a,b) \in (A \times B) \cup (A \times C)$, but not both.

Now, notice that $(A \times B)$ and $(A \times C)$ are disjoint. Assume otherwise. Assume $(A \times B) \cap (B \times C) \neq \emptyset$, and that $B \cap C = \emptyset$. If $(a,b) \in (A \times B) \cap (A \times C)$, then $(a,b) \in A \times B$ and $(a,b) \in A \times C$. That means $b \in B$ and $b \in C$, by definition of the Cartesian product, from which we can conclude $b \in B \cap C$, which is a contradiction. Thus,

$$(a,b) \in A \times (B \sqcup C) \text{ if and only if } (a,b) \in (A \times B) \sqcup (A \times C),$$

which by definition of set equality means

$$A \times (B \sqcup C) = (A \times B) \sqcup (A \times C),$$

as desired.

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4. <u>proof:</u> Assume $f: X \to Y$ is a map and let $\mathcal{B} \subset \mathcal{S}(Y)$. We will show that

$$x \in f^{-1}\Big(\bigcap_{B \in \mathcal{B}} B\Big) \text{ if and only if } x \in \bigcap_{B \in \mathcal{B}} f^{-1}(B),$$

which by definition, yields the set equality

$$f^{-1}\Big(\bigcap_{B \in \mathcal{B}} B\Big) = \bigcap_{B \in \mathcal{B}} f^{-1}(B).$$

Let $x \in X$.

$$x \in f^{-1}\Big(\bigcap_{B \in \mathcal{B}} B\Big) \iff f(x) \in \bigcap_{B \in \mathcal{B}} B,$$ by definition of the preimage,

$$\iff f(x) \in B \text{ for all } B \in \mathcal{B},$$ by definition of an arbitrary intersection of sets.

$\iff$ $x \in f^{-1}(B)$ for all $B \in \mathcal{B}$,

by definition of the preimage.

$\iff$ $x \in \bigcap_{B \in \mathcal{B}} f^{-1}(B)$, by definition of an arbitrary intersection of sets.

Thus, $x \in f^{-1}\left(\bigcap_{B \in \mathcal{B}} B\right)$ if and only if $x \in \bigcap_{B \in \mathcal{B}} f^{-1}(B)$,

which by definition of set equality means $f^{-1}\left(\bigcap_{B \in \mathcal{B}} B\right) = \bigcap_{B \in \mathcal{B}} f^{-1}(B)$,

as desired.

$\blacksquare$

5. <u>proof</u>: ① Assume $f: X \longrightarrow Y$ is a map of arbitrary sets X and Y, and let $A \subset X$. Let $x \in A$.

Then

$x \in A \implies f(x) \in f(A)$, by definition of of the image. (Notice that the converse need not hold!)

$\iff$ $f(x) = f(\tilde{x})$ for some $\tilde{x} \in A$, by the (full) definition of the preimage.

this is what the converse would tell us! We do not need it though, just need it to understand why our first implication has only one direction!

So,

$x \in A \implies f(x) \in f(A)$, by definition of the image.

$\iff x \in f^{-1}(f(A))$, by definition of the preimage.

Hence, by definition of subset, $A \subset f^{-1}(f(A))$.

② Assume $f: X \to Y$ and $B \subset Y$. Let $y \in Y$.

Then

$y \in f(f^{-1}(B)) \iff y = f(x)$ for some $x \in f^{-1}(B)$, by definition of the image,

$\iff y = f(x)$ for some $f(x) \in B$, by definition of the preimage,

$\implies y \in B$, since $y = f(x)$.

notice, $y \in B$
does not imply $y = f(x)$ for some $f(x) \in B$!

Thus, by definition of subset, $f(f^{-1}(B)) \subset B$, as desired.

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6. <u>proof</u>: ⓪ (We must first show f^{-1} is bijective!)

Assume $f: X \to Y$ is a bijective map, and let f^{-1} denote its inverse function, $f^{-1}: Y \to X$. First, observe that f^{-1} is injective. Let $y_1, y_2 \in Y$. If $f^{-1}(y_1) = f^{-1}(y_2)$, then by definition of the inverse function $f(f^{-1}(y_1)) = y_1$ and $f(f^{-1}(y_2)) = y_2$.

(Notice, we are just using one implication of the definition: $\underbrace{f^{-1}(y)}$ $= x$ if and only if $f(x) = y$.)

plug in!

notice, we might suspect we need to leverage the fact that f is injective, but we seemingly never do. This is <u>embedded</u> in the definition of the inverse function! If f weren't injective, the definition of f^{-1} wouldn't make sense!

This means $y_1 \overset{\text{def.!}}{=} f(f^{-1}(y_1)) \overset{\text{assumption!}}{=} f(f^{-1}(y_2)) \overset{\text{def!}}{=} y_2$.

Thus, we have shown if $f^{-1}(y_1) = f^{-1}(y_2)$, then $y_1 = y_2$, so f^{-1} is injective by definition.

Now we show that f^{-1} is surjective. Let $x \in X$. Then $f(x) = y$ for some $y \in Y$, since f is a map. By definition of the inverse function, $f^{-1}(y) = x$. Thus, there exists a y such that $f^{-1}(y) = x$. Since y was arbitrary (any $y \in Y$ works), this is the definition of surjectivity. Thus, f^{-1} is surjective.

Since f^{-1} is both injective and surjective, f^{-1} is bijective as desired.

① Let $f: X \to Y$ be a bijective map with $f^{-1}: Y \to X$ the inverse function. By ⓪, f^{-1} is bijective, so there exists an inverse of this inverse function, which we denote $(f^{-1})^{-1}$. $(f^{-1})^{-1}: X \to Y$ by definition of the inverse function, which means f and $(f^{-1})^{-1}$ have the same domain and codomain.

To show these maps are equal, by definition, we need only show that $(f^{-1})^{-1}(x) = f(x)$ for any $x \in X$. Using the definition of the inverse function, $(f^{-1})^{-1}(x) = y$ if and only if $f^{-1}(y) = x$. Applying the inverse function definition again, we see that $f^{-1}(y) = x$ if and only if $f(x) = y$, thus $f(x) = (f^{-1})^{-1}(x)$, which completes the proof.

(2) Let $f: X \to Y$ be bijective, and $f^{-1}: Y \to X$ be its bijective inverse (by (0)). Notice, by definition of the composition $f^{-1} \circ f : X \to X$. Since $i_X : X \to X$ is a map with the same domain and codomain, we only need to show that for any $x \in X$, $f^{-1} \circ f(x) = i_X(x)$ in order to conclude $f^{-1} \circ f = i_X$, using the definition of map equality.

First, notice $i_X(x) = x$, by definition of the identity map. Using the definition of composite maps, then the definition of the inverse function:

$$f^{-1} \circ f(x) = f^{-1}(f(x)) = x.$$

Thus, for any $x \in X$, $f^{-1} \circ f(x) = x = i_X$, as desired.

(3) Similar to above, let $f: X \to Y$ be bijective, and $f^{-1}: Y \to X$ be its bijective inverse. Notice, by definition of the composition, $f \circ f^{-1}: Y \to Y$. Since $i_Y: Y \to Y$ is a map with the same domain and codomain, we need only show for any $y \in Y$, $f \circ f^{-1}(y) = i_Y(y)$ in order to conclude $f \circ f^{-1} = i_Y$.

Notice that for any $y \in Y$, $i_Y(y) = y$. Using the definition of the composite map followed by the definition of the inverse function:

$$f \circ f^{-1}(y) = f(f^{-1}(y)) = y.$$

Thus, for any $y \in Y$, $f \circ f^{-1}(y) = y = i_Y(y)$, so the maps are equal by definition.

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