Homework 2 Answer Key

1. Provide a sketch and a proof.

proof: Assume a and b are negative, real numbers and that $a<b$. Since a is negative, if we multiply $a<b$ by a on both sides, we get $a^2>ab$. Similarly, since b is negative, if we multiply $a<b$ by b on both sides, we get $ab>b^2$. Thus, $a^2>ab$ and $ab>b^2$, in other words $a^2>ab>b^2$. This implies $a^2>b^2$, as desired. ▨

2. Provide a sketch and a proof for each statement.

(a) proof: Assume n is an even prime. Then n is divisible by 2, and the only divisors of n are 1 and itself. Thus, $n=2$. ▨

(b) proof: Assume n is even. Then $n=2k$ for some integer k. That means

$$n^3 = (2k)^3 = 2^3k^3 = 2(4k^3)$$

so we see that n^3 is divisible by 2. Thus, n^3 is even, as desired.

$\blacksquare$

3. Provide a sketch and a proof.

proof: Assume c and a are prime numbers such that $c \neq a$. Observe

$$(c-a)(c^2+ca+a^2) = (c-a)c^2 + (c-a)ca + (c-a)a^2$$
$$= c^3 - ac^2 + c^2a - ca^2 + ca^2 - a^3$$
$$= c^3 - a^3.$$

Now notice

$$|c^3 - a^3| = \begin{cases} c^3 - a^3 & \text{if } c^3 - a^3 > 0 \\ -(c^3 - a^3) & \text{if } c^3 - a^3 < 0, \end{cases}$$

by definition of the absolute value.

If $c^3 - a^3 > 0$, since $c^3 - a^3 = (c-a)(c^2+ca+a^2)$, we see that either both $(c-a)$ and c^2+ca+a^2 is positive or both negative. First note that if $\hat{a} > 0$, $\hat{b} > 0$, then $\hat{a}\hat{b} > 0$ by Elementary property 10. Now, assuming "a" and "b" are positive in Elementary properties 4 and 5, the consequence follows.

In either scenario,

$$|c-a|\,|c^2+ac+a^2| = \begin{cases} (c-a)(c^2+ac+a^2), & \text{both positive} \\ -(c-a)\cdot -(c^2+ac+a^2), & \text{both negative} \end{cases}$$

$$= \begin{cases} (c-a)(c^2+ac+a^2), & \text{both positive} \\ (c-a)\cdot(c^2+ac+a^2), & \text{both negative} \end{cases}$$

$$= (c-a)(c^2+ac+a^2)$$

$$= c^3-a^3$$

$$= |c^3-a^3|, \quad \text{since we assumed } c^3-a^3>0.$$

Now assume $c^3-a^3<0$. As before, since $c^3-a^3=(c-a)(c^2+ca+a^2)$, we see that either $c-a<0$ or $c^2+ca+a^2<0$ (by applying Elementary property 5 with the fact that for $\tilde{a}>0, \tilde{b}>0$, $\tilde{a}\tilde{b}>0$ by Elementary property 10).

In either scenario,

$$|c-a|\,|c^2+ac+a^2| = \begin{cases} -(c-a)\cdot(c^2+ac+a^2) & \text{if } (c-a)<0 \\ (c-a)\cdot -(c^2+ac+a^2) & \text{if } (c^2+ac+a^2)<0 \end{cases}$$

(*) you can eliminate this possibility by using EPI 15 and EPI 12.

$$= -(c-a)(c^2+ac+a^2)$$

$$= -(c^3-a^3)$$

$$= |c^3-a^3|,$$ since we assumed $c^3-a^3<0$.

Since $c \neq a$, $c-a \neq 0$ and $c^3-a^3 \neq 0$, so we need not consider the case where $c^3-a^3=0$. Hence, we have shown

$$|c^3-a^3| = |c-a|\cdot|c^2+ca+c^2|.$$

Now, since $c \neq a$, $c-a=0$, so either $c-a<0$ or $c-a>0$. Since integers are closed under addition and scalar multiplication (axiom), we know that $c-a=k$ for some integer $k \neq 0$. Thus,

$$|c-a| = \begin{cases} k, & k>0 \\ -k, & k<0 \end{cases} \geq 1.$$

Now consider $|c^2 + ca + a^2|$. Since a, c are prime numbers, $a > 1$ and $c > 1$ by definition. In other words $a \geq 2$ and $c \geq 2$. This means

$$a^2 \geq 2a \geq 4 \quad \text{by Elem. prop. 10.}$$

and

$$c^2 \geq 2a \geq 4 \quad \text{by Elem. prop. 10.}$$

and

$$ca \geq 2a \geq 4 \quad \text{by Elem. prop. 10.}$$

In other words

$$\begin{aligned} |c^2 + ca + a^2| &\geq 4 + ca + a^2 \\ &\geq 4 + 4 + a^2 \\ &\geq 4 + 4 + 4 = 12, \end{aligned}$$

by successive applications of Elementary Property 9.

So $|c^2 + ca + a^2| \geq 12$, by definition of the absolute value.

Thus,

$$\begin{aligned} |c^3 - a^3| &= |c-a|\,|c^2 - ca - a^2| \\ &\geq 1 \cdot 12, \end{aligned}$$

by Elementary Property 13, as desired. $\boxtimes$

5. Provide a sketch and a proof.

proof: Assume a divides b and b divides c.

Then $b = a \cdot n$ for some integer n and

$c = b \cdot m$ for some integer m.

Now, observe that

$$c = b \cdot m = a \cdot n \cdot m$$
$$= a(nm),$$

so we can conclude a divides c, by definition. $\blacksquare$

6. Provide a proof only.

proof: Assume for the sake of contradiction that $r^2 = 2$ and r is rational. Since r is rational, we may write $r = \frac{m}{n}$. In addition, we may assume m and n have no common divisors, otherwise we could divide them out.

Since $r^2 = 2$, we have that $r^2 = \frac{m^2}{n^2} = 2$, which means $2n^2 = m^2$. Thus, 2 is a divisor of m^2 (so m^2 is even).

Question 4 tells us that m^2 is odd if and only if m is odd. By taking the contrapositive of both implications, we are led to the following fact: m^2 is even if and only if m is even. Thus, since m^2 is even, we know m is even.

This means 2 is a divisor of m, so we may write $m = 2k$ for some integer k. Since $2n^2 = m^2$, we have the following:

$$2n^2 = m^2 = (2k)^2 = 4k^2 = 2(2k^2).$$

So, $n^2 = 2k^2$, and we see that 2 is also a divisor of n^2. Hence, as before, n^2 is even, so we may conclude by Question 4 that n is even.

However, that means that both m and n are even, that both m and n share 2 as a divisors. This is a contradiction since we picked m and n to have no common divisors. $\blacksquare$

7. Provide a sketch and a proof.

We prove the following statement:

"If m, n are even integers, then $m+n$ is an even integer."

proof: Assume for the sake of contradiction that m and n are even integers and $m+n$ is odd. Then we may write $m=2k$ for some integer k and $n=2\ell$ for some integer ℓ. Computing, we find $m+n = 2k+2\ell = 2(k+\ell)$. Thus, we see $m+n$ is even, which contradicts our assumption that $m+n$ is odd.

$\boxtimes$

Bonus: This proof is bad form for the following reason. We make the additional assumption that $m+n$ is odd (for a contradiction proof), then we prove directly that $m+n$ is even, then claim a contradiction with our initial assumption. If we remove this assumption, and the last line claiming a contradiction, we have a direct proof. (Contradiction was unnecessary!)